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CONTENTS

The Theory of the Binary Octavic. By ANNA MARIE WHELAN, . . 73

The Borel Summability of Fourier Series. By M. H. STONE, . . . 101

Self Dual Space Curves. By TEMPLE RICE HOLLCROFT, ... 113

Expansions in Terms of Certain Polynomials Connected with the Gamma-Function. By BORDEN PARKER HOOVER, . . . 125

On the Value of the Napierian Base. By DERRICK HENRY LEHMER, . 139

On Differential Inversive Geometry. By FRANK MORLEY, ... 144

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The Theory of the Binary Octavic.

By Anna Marie Whelan.

Introduction.

The theory of the binary octavic is the theory of eight numbers. We shall denote these numbers by the symbols 1, 2, $\cdots$, 8. Out of these we may form products of differences of the type:

$$\pi = 12 \cdot 34 \cdot 56 \cdot 78$$

where 12 denotes the excess of quantity 1 over 2. The number of such products π is 105, for in any such product any number, say 1, occurs combined with 7 others giving us differences of the type 12, and each such difference 12 occurs in fifteen products, the fifteen products of differences of six points.

Eight points may be paired in 28 ways. Any product such as $12 \cdot 34 \cdot 56 \cdot 78$ involves four differences, or pairs, so our 105 products fall into sets of seven such that no two products of the set contain a common pair. There is more than one type of such sets as we may see by considering the two sets H and H'.

	H		H'
π_1	$12 \cdot 34 \cdot 56 \cdot 78$	p_1	$12 \cdot 34 \cdot 56 \cdot 78$
π_2	$13 \cdot 24 \cdot 75 \cdot 86$	p_2	$23 \cdot 45 \cdot 67 \cdot 18$
π_3	$14 \cdot 32 \cdot 67 \cdot 85$	p_3	$13 \cdot 24 \cdot 57 \cdot 68$
π_4	$15 \cdot 62 \cdot 37 \cdot 84$	p_4	$17 \cdot 28 \cdot 35 \cdot 46$
π_5	$16 \cdot 25 \cdot 74 \cdot 83$	p_5	$14 \cdot 27 \cdot 36 \cdot 58$
π_6	$17 \cdot 82 \cdot 46 \cdot 53$	p_6	$16 \cdot 25 \cdot 38 \cdot 47$
π_7	$18 \cdot 27 \cdot 36 \cdot 45$	p_7	$15 \cdot 26 \cdot 37 \cdot 48$

Both H and H' contain all 28 pairs, but they are very different in type. In H, π_1 has a pair of factors $12 \cdot 34$ whose four elements occur as a pair of factors again in π_2 as $13 \cdot 24$, and again in π_3 as $14 \cdot 32$. In general any two products of H pick out a third product of H such that the three products have this property. This is not true of H'.

We shall refer to the sets of 7 products as heptads. We are interested

only in heptads of the H type. Let us determine the number of these. To do this we consider the possible heptads built up out of the product

$$\pi_1 = 12\cdot34\cdot56\cdot78.$$

We note that this product picks out 3 pairs of factors viz.: 12·34; 12·56; 12·78. Hence our heptad must contain the pairs:

12·34	12·56	12·78
13·24	15·26	17·28
14·32	16·25	18·27.

So, starting with $\pi_1 = 12\cdot34\cdot56\cdot78$, we can fill second place with

$$\pi_2 = 13\cdot24\cdot57\cdot68 \text{ or } 13\cdot24\cdot58\cdot67.$$

Choosing the first of these we must have:

$$\pi_3 = 14\cdot23\cdot58\cdot76.$$

The pair 15·26 can occur in 15·26·34·78, but this product is ruled out since 34·78 already occurs. Moreover the product 15·26·37·48 is also ruled out, since in the second product 1, 5, 3 occur combined with 8. Consequently we have only one choice, viz., 15·26·38·47, for fourth place. Likewise the remaining three products are uniquely determined. Hence our heptad is completely determined by two products. We note that each product occurs in two heptads giving us 210 products forming sets of seven. Consequently the number of heptads is 30. These heptads may be called the Noether-Coble Heptads since they were first indicated by Noether and were first given by Coble.

§ 1. *Construction of the Thirty Noether-Coble Heptads.*

We have constructed one heptad:

$$\begin{aligned}\sigma_1 = {} & 12\cdot34\cdot56\cdot78\\ & 13\cdot24\cdot75\cdot86\\ & 14\cdot32\cdot67\cdot85\\ & 15\cdot62\cdot37\cdot84\\ & 16\cdot25\cdot74\cdot83\\ & 17\cdot82\cdot46\cdot53\\ & 18\cdot27\cdot36\cdot45.\end{aligned}$$

Denoting the 30 heptads by

$$\sigma_1, \sigma_2, \cdot\cdot\cdot, \sigma_{15}, \quad s_1, s_2, \cdot\cdot\cdot, s_{15},$$

we may, by applying the following substitutions, obtain the complete set.

$(12)\sigma_1 = -s_1$	$(16)\sigma_3 = -s_{10}$	$(18)s_1 = -\sigma_6$
$(13)\sigma_1 = -s_2$	$(17)\sigma_3 = -s_{11}$	$(15)s_2 = -\sigma_9$
$(14)\sigma_1 = -s_3$	$(18)\sigma_3 = -s_9$	$(16)s_2 = -\sigma_{10}$
$(15)\sigma_1 = -s_4$	$(15)\sigma_2 = -s_{12}$	$(17)s_2 = -\sigma_8$
$(16)\sigma_1 = -s_5$	$(16)\sigma_2 = -s_{15}$	$(18)s_2 = -\sigma_{11}$
$(17)\sigma_1 = -s_6$	$(17)\sigma_2 = -s_{13}$	$(15)s_3 = -\sigma_{12}$
$(18)\sigma_1 = -s_7$	$(18)\sigma_2 = -s_{14}$	$(16)s_3 = -\sigma_{14}$
$(567)\sigma_1 = \sigma_2$	$(15)s_1 = -\sigma_5$	$(17)s_3 = -\sigma_{15}$
$(567)\sigma_2 = \sigma_3$	$(16)s_1 = -\sigma_4$	$(18)s_3 = -\sigma_{13}$
$(15)\sigma_3 = -s_8$	$(17)s_1 = -\sigma_7$	

§ 2. *Group of the Heptads.*

These thirty heptads admit the group $G_{8!}$. They divide into two sets of 15 heptads

$$\Sigma = \sigma_1, \sigma_1, \cdots, \sigma_{15},$$
$$S = s_1, s_2, \cdots, s_{15},$$

such that Σ and S are unaltered under the even substitutions of the $G_{8!}$, while Σ is sent into $-S$ under the odd substitutions of $G_{8!}$. This fact may be proved by applying the generators of $G_{8!}/_2$ to the table of heptads which we include:

THE NOETHER-COBLE HEPTADS.

σ_1	12·34·56·78	σ_6	12·36·45·78	σ_{11}	17·65·23·48
	13·24·75·86		13·62·47·85		16·57·24·83
	14·32·67·85		16·23·57·48		15·76·28·34
	15·62·37·84		14·52·37·86		12·37·64·85
	16·25·74·83		15·24·38·67		13·72·68·54
	17·82·46·53		17·34·56·82		14·87·62·35
	18·27·36·45		18·27·35·64		18·74·63·52
σ_2	12·34·67·58	σ_7	18·27·43·65	σ_{12}	13·47·28·65
	13·42·56·78		17·82·45·36		17·34·25·86
	14·23·57·86		12·78·46·53		14·73·26·58
	15·82·63·47		13·84·25·76		12·83·46·57
	16·48·27·35		14·38·26·57		18·32·45·76
	18·64·25·73		15·86·23·74		15·36·48·72
	17·62·45·83		16·58·24·37		16·53·42·87

σ_3	12·34·68·75	σ_8	12·38·45·67	σ_{13}	16·57·28·34
	13·24·58·76		15·24·37·86		17·65·24·83
	14·32·65·78		14·52·36·78		15·76·23·48
	15·83·27·64		13·75·28·64		12·86·53·47
	16·82·37·54		17·53·26·48		18·62·54·73
	17·84·36·52		16·58·27·43		14·63·58·72
	18·35·26·47		18·65·23·74		13·46·52·87
σ_4	12·38·65·74	σ_9	12·36·47·85	σ_{14}	14·52·38·67
	13·82·67·45		16·23·45·78		15·24·36·78
	14·86·27·35		13·62·48·57		12·45·37·86
	15·78·62·34		14·72·38·56		13·84·72·56
	16·48·25·73		17·24·35·68		18·43·57·26
	17·85·42·63		15·28·37·64		16·74·53·82
	18·64·23·57		18·52·34·76		17·46·58·23
σ_5	12·73·56·48	σ_{10}	17·34·26·58	σ_{15}	12·36·48·57
	13·58·27·64		13·47·25·86		16·23·47·85
	14·36·57·82		14·73·28·65		13·62·45·78
	15·83·26·47		12·67·35·84		14·82·35·76
	16·43·52·78		16·72·38·45		18·24·37·65
	17·32·54·86		15·87·32·64		15·72·34·86
	18·35·24·76		18·75·36·42		17·25·38·64
s_1	13·24·58·76	s_6	12·36·45·78	s_{11}	15·72·34·86
	12·34·56·78		13·62·48·57		17·84·36·52
	14·23·86·57		16·23·47·85		12·83·46·57
	15·83·26·47		14·52·67·38		13·82·67·45
	16·48·25·73		15·24·37·86		14·87·62·35
	17·82·45·36		18·27·43·65		16·58·24·37
	18·27·35·64		17·82·46·53		18·65·23·74
s_2	12·34·58·67	s_7	17·34·56·82	s_{12}	18·52·34·76
	13·24·75·86		18·27·36·45		15·63·47·82
	14·32·78·65		12·78·46·53		12·37·64·85
	15·28·37·64		13·47·25·86		13·84·72·56
	16·72·38·45		14·73·26·58		14·38·26·57
	17·53·26·48		15·76·23·48		16·53·42·87
	18·74·63·52		16·57·24·83		17·32·54·86

s_3	12·34·68·75	s_8	12·36·48·57	s_{13}	18·24·37·65
	13·42·56·78		16·23·45·78		14·36·57·82
	14·32·67·85		13·62·47·85		12·67·35·84
	15·36·48·72		14·73·28·65		13·46·52·87
	16·74·53·82		18·35·24·76		17·62·45·83
	17·25·38·64		15·83·27·64		15·86·23·74
	18·62·54·73		17·34·25·86		16·58·27·43
s_4	17·24·35·68	s_9	15·24·38·67	s_{14}	12·36·47·85
	14·63·58·72		14·52·36·78		16·23·48·57
	12·38·65·74		12·45·37·86		13·62·45·78
	13·75·28·64		13·58·27·64		14·27·35·86
	18·67·54·32		18·35·26·47		17·24·83·65
	15·37·84·62		16·57·28·34		15·28·34·76
	16·43·52·78		17·65·23·48		18·52·37·64
s_5	14·82·35·76	s_{10}	14·72·38·56	s_{15}	14·52·37·86
	18·75·36·42		12·86·53·47		15·24·36·78
	12·73·56·48		17·85·42·63		13·47·28·65
	13·72·68·54		13·84·25·76		17·34·26·58
	15·78·62·34		18·43·57·26		12·45·38·67
	16·74·83·25		16·82·37·54		18·64·23·57
	17·46·58·23				16·48·27·35

In constructing the heptads we have attached a sign to each product by considering the product 12·34·56·78 as positive. Every other product is positive or negative, according as it is obtained from 12·34·56·78 by an even or an odd substitution. In the table every product is positive.

Every heptad may be regarded as an Abelian group of order eight. If we consider the products 12·34·56·78 as substitutions of order 2 and add the identity we see that σ_i forms an Abelian group of order 8 under which the heptad σ_i is invariant. If we let A, B, C denote respectively the substitutions: 12·34·56·78; 13·24·57·68; 15·26·37·48 we may write the substitutions of the group in the form:

$$1,\ A,\ B,\ AB,\ C,\ AC,\ BC,\ ABC.$$

§ 3. *Triad System on the Heptads.*

Considering the table, we note that the pair 12·34 occurs in 6 heptads $\sigma_1\sigma_2\sigma_3$, $s_1s_2s_3$. Restricting ourselves to the σ-heptads, we see that the

pairing (12·34) picks out a triad $\sigma_1\sigma_2\sigma_3$. There are 35 divisions of the type (12·34). Consequently we have 35 triads of the type $\sigma_1\sigma_2\sigma_3$. These are:

$\sigma_1\sigma_2\sigma_3$	(1234)	$\sigma_5\sigma_{10}\sigma_{15}$	(1248)
$\sigma_1\sigma_4\sigma_5$	(1256)	$\sigma_5\sigma_{11}\sigma_{14}$	(1732)
$\sigma_1\sigma_6\sigma_7$	(1278)	$\sigma_6\sigma_8\sigma_{14}$	(1524)
$\sigma_1\sigma_8\sigma_9$	(1357)	$\sigma_6\sigma_9\sigma_{15}$	(1236)
$\sigma_1\sigma_{10}\sigma_{11}$	(1386)	$\sigma_3\sigma_{12}\sigma_{15}$	(1527)
$\sigma_1\sigma_{12}\sigma_{13}$	(1485)	$\sigma_3\sigma_{13}\sigma_{14}$	(1826)
$\sigma_1\sigma_{14}\sigma_{15}$	(1467)	$\sigma_4\sigma_8\sigma_{12}$	(1823)
$\sigma_2\sigma_4\sigma_6$	(1486)	$\sigma_4\sigma_9\sigma_{13}$	(1427)
$\sigma_2\sigma_5\sigma_7$	(1457)	$\sigma_4\sigma_{10}\sigma_{14}$	(1578)
$\sigma_2\sigma_8\sigma_{10}$	(1267)	$\sigma_4\sigma_{11}\sigma_{15}$	(1345)
$\sigma_2\sigma_9\sigma_{11}$	(1258)	$\sigma_5\sigma_8\sigma_{13}$	(1364)
$\sigma_2\sigma_{12}\sigma_{14}$	(1635)	$\sigma_6\sigma_{10}\sigma_{12}$	(1437)
$\sigma_2\sigma_{13}\sigma_{15}$	(1738)	$\sigma_6\sigma_{11}\sigma_{13}$	(1756)
$\sigma_3\sigma_4\sigma_7$	(1637)	$\sigma_7\sigma_8\sigma_{15}$	(1856)
$\sigma_3\sigma_5\sigma_6$	(1358)	$\sigma_7\sigma_{10}\sigma_{13}$	(1235)
$\sigma_3\sigma_8\sigma_{11}$	(1478)	$\sigma_7\sigma_9\sigma_{14}$	(1348)
$\sigma_3\sigma_9\sigma_{10}$	(1564)	$\sigma_7\sigma_{11}\sigma_{12}$	(1624)
$\sigma_5\sigma_9\sigma_{12}$	(1678)		

This triad system belongs to the class which has a "head." This head is composed of the following 7 triads:

$$\sigma_2\sigma_4\sigma_6,\quad \sigma_2\sigma_8\sigma_{10},\quad \sigma_2\sigma_{12}\sigma_{14},\quad \sigma_4\sigma_8\sigma_{12},$$
$$\sigma_4\sigma_{10}\sigma_{14},\quad \sigma_6\sigma_8\sigma_{14},\quad \sigma_6\sigma_{10}\sigma_{12}.$$

Every triple of this system is of index 1^{12} and admits the group $G_{8!/2}$. It is the Kirkman Triad System.*

We will now denote by σ_i and s_i the sum of the seven products in the heptads σ_i, s_i. We note that σ_1 is sent into $-s_1$ by the interchange (12) whence $\sigma_1 + s_1$ is sent into $-(\sigma_1 + s_1)$. Consequently $\sigma_1 + s_1$ contains the factor 12. Likewise it contains the factors 34·56·78. Hence.

$$\sigma_1 + s_1 = \kappa \cdot 12\cdot 34\cdot 56\cdot 78.$$

To determine the multiplier κ, we note that

$$12\cdot 34 + 13\cdot 42 + 14\cdot 23 = 0,$$
$$56\cdot 78 + 57\cdot 86 + 58\cdot 67 = 0, \text{ etc.}$$

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* See White, Cole, Cummings, *Memoirs of the National Academy*, Vol. XIV, Second Memoir, p. 77.

$$\begin{aligned} \text{Now } \sigma_1 = {} & 12\cdot34\cdot56\cdot78 \ (\equiv\pi_1) & S_1 = {} & 12\cdot34\cdot56\cdot78 \ (\equiv\pi_1) \\ & +13\cdot24\cdot75\cdot86 \ (\equiv\pi_2) & & +13\cdot24\cdot58\cdot76 \ (\equiv\pi_2') \\ & +14\cdot32\cdot67\cdot85 \ (\equiv\pi_3) & & +14\cdot23\cdot86\cdot57 \ (\equiv\pi_3') \\ & +15\cdot62\cdot37\cdot84 \ (\equiv\pi_4) & & +15\cdot83\cdot26\cdot47 \ (\equiv\pi'_4) \\ & +16\cdot25\cdot74\cdot83 \ (\equiv\pi_5) & & +16\cdot48\cdot25\cdot73 \ (\equiv\pi'_5) \\ & +17\cdot82\cdot46\cdot53 \ (\equiv\pi_6) & & +17\cdot82\cdot45\cdot36 \ (\equiv\pi_6') \\ & +18\cdot27\cdot36\cdot45 \ (\equiv\pi_7) & & +18\cdot27\cdot35\cdot64 \ (\equiv\pi_7'); \end{aligned}$$

so that

$$\begin{aligned} &\pi_2 + \pi_3' = (13\cdot24 + 14\cdot32)75\cdot86 = 12\cdot34\cdot75\cdot86 \\ &\pi_3 + \pi_2' = (14\cdot32 + 13\cdot24)67\cdot85 = 12\cdot34\cdot67\cdot85 \\ &\pi_2 + \pi_3 + \pi_2' + \pi_3' = 12\cdot34\cdot56\cdot78 \\ &\pi_4 + \pi_5' = 12\cdot56\cdot73\cdot84 \\ &\pi_5 + \pi_4' = 12\cdot56\cdot47\cdot83 \\ &\pi_4 + \pi_5 + \pi_4' + \pi_5' = 12\cdot34\cdot56\cdot78. \end{aligned}$$

Whence $\sigma_1 + s_1 = 5\cdot12\cdot34\cdot56\cdot78$.

We have therefore proved that the sum of two heptads having a common product is equal to 5 times that common product. The seven products of every heptad occur individually in 7 other heptads. We shall call these 7 heptads the correlates of the first. Thus σ_1 has for correlates $s_1s_2s_3s_4s_5s_6s_7$. Any heptad may be expressed linearly in terms of its correlates. Consider the heptad σ_1:

$$\begin{aligned} &\sigma_1 + s_1 = 5\pi_1 \qquad && \sigma_1 + s_4 = 5\pi_4 \qquad && \sigma_1 + s_6 = 5\pi_6 \\ &\sigma_1 + s_2 = 5\pi_2 && \sigma_1 + s_5 = 5\pi_5 && \sigma_1 + s_7 = 5\pi_7 \\ &\sigma_1 + s_3 = 5\pi_3 \end{aligned}$$

Adding we get:

$$7\sigma_1 + s_1 + s_2 + \cdots s_7 = 5\sigma_1$$

whence
$$\sigma_1 = -\tfrac{1}{2}(s_1 + s_2 + \cdots s_7).$$

Thus we see that every heptad is equal to minus one-half the sum of its seven correlates.

Consider now the six heptads on the pairing (1234). These are σ_1, σ_2, σ_3, s_1, s_2, s_3.

$$\begin{aligned} \sigma_1 + s_1 + \sigma_2 + s_2 + \sigma_3 + s_3 &= 5(12\cdot34\cdot56\cdot78 + 12\cdot34\cdot67\cdot58 + 12\cdot34\cdot68\cdot75) \\ &= 5\cdot12\cdot34(56\cdot78 + 58\cdot67 + 57\cdot86) \\ &= 0. \end{aligned}$$

Consequently we have 35 identities among the heptads, six at a time. They are:

$$(1)\quad \sigma_1+\sigma_2+\sigma_3+s_1+s_2+s_3=0$$
$$(2)\quad \sigma_1+\sigma_4+\sigma_5+s_1+s_4+s_5=0$$
$$(3)\quad \sigma_1+\sigma_6+\sigma_7+s_1+s_6+s_7=0$$
$$(4)\quad \sigma_1+\sigma_8+\sigma_9+s_2+s_4+s_6=0$$
$$(5)\quad \sigma_1+\sigma_{10}+\sigma_{11}+s_2+s_5+s_7=0$$
$$(6)\quad \sigma_1+\sigma_{12}+\sigma_{13}+s_3+s_4+s_7=0$$
$$(7)\quad \sigma_1+\sigma_{14}+\sigma_{15}+s_3+s_5+s_6=0$$
$$(8)\quad \sigma_2+\sigma_4+\sigma_6+s_1+s_{14}+s_{15}=0$$
$$(9)\quad \sigma_2+\sigma_5+\sigma_7+s_1+s_{12}+s_{13}=0$$
$$(10)\quad \sigma_2+\sigma_8+\sigma_{10}+s_2+s_{13}+s_{15}=0$$
$$(11)\quad \sigma_2+\sigma_9+\sigma_{11}+s_2+s_{12}+s_{14}=0$$
$$(12)\quad \sigma_2+\sigma_{12}+\sigma_{14}+s_3+s_{12}+s_{15}=0$$
$$(13)\quad \sigma_2+\sigma_{13}+\sigma_{15}+s_3+s_{13}+s_{14}=0$$
$$(14)\quad \sigma_3+\sigma_4+\sigma_7+s_1+s_{10}+s_{11}=0$$
$$(15)\quad \sigma_3+\sigma_5+\sigma_6+s_1+s_8+s_9=0$$
$$(16)\quad \sigma_3+\sigma_8+\sigma_{11}+s_2+s_9+s_{11}=0$$
$$(17)\quad \sigma_3+\sigma_9+\sigma_{10}+s_2+s_8+s_{10}=0$$
$$(18)\quad \sigma_3+\sigma_{12}+\sigma_{15}+s_3+s_8+s_{11}=0$$
$$(19)\quad \sigma_3+\sigma_{13}+\sigma_{14}+s_3+s_9+s_{10}=0$$
$$(20)\quad \sigma_4+\sigma_8+\sigma_{12}+s_4+s_{11}+s_{15}=0$$
$$(21)\quad \sigma_4+\sigma_9+\sigma_{13}+s_4+s_{10}+s_{14}=0$$
$$(22)\quad \sigma_4+\sigma_{10}+\sigma_{14}+s_5+s_{10}+s_{15}=0$$
$$(23)\quad \sigma_4+\sigma_{11}+\sigma_{15}+s_5+s_{11}+s_{14}=0$$
$$(24)\quad \sigma_5+\sigma_8+\sigma_{13}+s_4+s_9+s_{13}=0$$
$$(25)\quad \sigma_5+\sigma_9+\sigma_{12}+s_4+s_8+s_{12}=0$$
$$(26)\quad \sigma_5+\sigma_{10}+\sigma_{15}+s_5+s_8+s_{13}=0$$
$$(27)\quad \sigma_5+\sigma_{11}+\sigma_{14}+s_5+s_9+s_{12}=0$$
$$(28)\quad \sigma_6+\sigma_8+\sigma_{14}+s_6+s_9+s_{15}=0$$
$$(29)\quad \sigma_6+\sigma_9+\sigma_{15}+s_6+s_8+s_{14}=0$$
$$(30)\quad \sigma_6+\sigma_{10}+\sigma_{12}+s_7+s_8+s_{15}=0$$
$$(31)\quad \sigma_6+\sigma_{11}+\sigma_{13}+s_7+s_9+s_{14}=0$$
$$(32)\quad \sigma_7+\sigma_8+\sigma_{15}+s_6+s_{11}+s_{13}=0$$
$$(33)\quad \sigma_7+\sigma_{10}+\sigma_{13}+s_7+s_{10}+s_{13}=0$$
$$(34)\quad \sigma_7+\sigma_{11}+\sigma_{12}+s_7+s_{11}+s_{12}=0$$
$$(35)\quad \sigma_7+\sigma_9+\sigma_{14}+s_6+s_{10}+s_{12}=0.$$

Each heptad occurs 7 times in these equations. Adding we get:

$$7(\sigma_1 + \sigma_2 + \cdots s_1 + \cdots s_{15}) = 0.$$

Now every product of the 105 is contained in both the σ- and s-heptads. Consequently

$$\Sigma\, \sigma_i = \Sigma\, s_i = \overset{105}{\Sigma}\, 12\cdot 34\cdot 56\cdot 78 = 0,$$

a result, which we may obtain in a different way as follows:

Consider the summation

$$\overset{105}{\Sigma}\, \pi_i^{2i+1} = \Sigma\, (12\cdot 34\cdot 56\cdot 78)^{2i+1}.$$

An odd interchange like (12) sends each term into the negation of some other term. Consequently 12 is a factor of the sum. Likewise all 28 differences 12 are factors of the sum. Hence for $i = 0$, 1, or 2, $\overset{105}{\Sigma}\, \pi_i^{2i+1} = 0$, since it is not of high enough degree to contain these 28 factors. For $i = 3$, $\Sigma\, \pi_i^{2i+1} = k\, 12\cdot 34 \cdots 78$. The same argument gives

$$\Sigma\, \sigma_i^{2i+1} + \Sigma\, s_i^{2i+1} = 0, \text{ for } i = 0, 1 \text{ or } 2,$$

but $\quad \Sigma\, \sigma_i^7 + \Sigma\, s_i^7 = 2\mu\;\, 12\cdot 13\cdot 14 \cdots 78.$

Also $\quad \Sigma\, \sigma_i^{2i} - \Sigma\, s_i^{2i} = 0$, for $i = 1, 2, 3$.

$\Sigma\, \sigma_i^8 - \Sigma\, s_i^8$ is of high enough degree to contain the product of the 28 differences, but it would have to be equal to this product multiplied by a factor of degree one, which is unaltered by any permutation of the roots. The octavic has no linear invariant. Consequently: $\Sigma\, \sigma_i^8 - \Sigma\, s_i^8 = 0$.

§ 4. *The Complete System of Invariants of the Octavic.*

The octavic has nine fundamental invariants,* one of each order from 2 to 10 inclusive. The functions

$$\Sigma\, \sigma_i^n + \Sigma\, s_i^n \text{ when } n \text{ is even, and}$$
$$\Sigma\, \sigma_i^n - \Sigma\, s_i^n \text{ when } n \text{ is odd,}$$

are unaltered by any permutations of the roots of the octavic and so are rational invariants of the octavic. We may therefore write as fundamental invariants:

$$\begin{array}{ll}
\Sigma\, \sigma_i^2 + \Sigma\, s_i^2 = 2\,\Sigma\, \sigma_i^2 = 2I_2 & \Sigma\, \sigma_i^7 - \Sigma\, s_i^7 = 2\,\Sigma\, \sigma_i^7 = 2I_7 \\
\Sigma\, \sigma_i^3 - \Sigma\, s_i^3 = 2\,\Sigma\, \sigma_i^3 = 2I_3 & \Sigma\, \sigma_i^8 + \Sigma\, s_i^8 = 2\,\Sigma\, \sigma_i^8 = 2I_8 \\
\Sigma\, \sigma_i^4 + \Sigma\, s_i^4 = 2\,\Sigma\, \sigma_i^4 = 2I_4 & \Sigma\, \sigma_i^9 - \Sigma\, s_i^9 = \qquad\quad 2I_9 \\
\Sigma\, \sigma_i^5 - \Sigma\, s_i^5 = 2\,\Sigma\, \sigma_i^5 = 2I_5 & \Sigma\, \sigma_i^{10} + \Sigma\, s_i^{10} = \qquad 2I_{10}. \\
\Sigma\, \sigma_i^6 + \Sigma\, s_i^6 = 2\,\Sigma\, \sigma_i^6 = 2I_6 &
\end{array}$$

---

* Pascal, *Repertorium der Höheren Mathematik*, Vol. I, ed. 1900.

This is the complete system for the octavic. We will find it convenient to have the following facts in parallel arrangement.

$$
\begin{array}{ll}
\Sigma\sigma_1^2 + \Sigma s_1^2 = 2I_2 & \Sigma\sigma_1^2 - \Sigma s_1^2 = 0 \\
\Sigma\sigma_1^3 - \Sigma s_1^3 = 2I_3 & \Sigma\sigma_1^3 + \Sigma s_1^3 = 0 \\
\Sigma\sigma_1^4 + \Sigma s_1^4 = 2I_4 & \Sigma\sigma_1^4 - \Sigma s_1^4 = 0 \\
\Sigma\sigma_1^5 - \Sigma s_1^5 = 2I_5 & \Sigma\sigma_1^5 + \Sigma s_1^5 = 0 \\
\Sigma\sigma_1^6 + \Sigma s_1^6 = 2I_6 & \Sigma\sigma_1^6 - \Sigma s_1^6 = 0 \\
\Sigma\sigma_1^7 - \Sigma s_1^7 = 2I_7 & \Sigma\sigma_1^7 + \Sigma s_1^7 = 2\mu\sqrt{\Delta} \\
\Sigma\sigma_1^8 + \Sigma s_1^8 = 2I_8 & \Sigma\sigma_1^8 - \Sigma s_1^8 = 0 \\
\Sigma\sigma_1^9 - \Sigma s_1^9 = 2I_9 & \Sigma\sigma_1^9 + \Sigma s_1^9 = 2\lambda_1\sqrt{\Delta}\, I_2 \\
\Sigma\sigma_1^{10} + \Sigma s_1^{10} = 2I_{10} & \Sigma\sigma_1^{10} - \Sigma s_1^{10} = 2\lambda_2\sqrt{\Delta}\, I_3 \\
\Sigma\sigma_1^{11} - \Sigma s_1^{11} = 2I_{11} & \Sigma\sigma_1^{11} + \Sigma s_1^{11} = 2\lambda_3\sqrt{\Delta}\, R_4 \\
\Sigma\sigma_1^{12} + \Sigma s_1^{12} = 2I_{12} & \Sigma\sigma_1^{12} - \Sigma s_1^{12} = 2\lambda_4\sqrt{\Delta}\, R_5 \\
\Sigma\sigma_1^{13} - \Sigma s_1^{13} = 2I_{13} & \Sigma\sigma_1^{13} + \Sigma s_1^{13} = 2\lambda_5\sqrt{\Delta}\, R_6 \\
\Sigma\sigma_1^{14} + \Sigma s_1^{14} = 2I_{14} & \Sigma\sigma_1^{14} - \Sigma s_1^{14} = 2\lambda_6\sqrt{\Delta}\, R_7 \\
\Sigma\sigma_1^{15} - \Sigma s_1^{15} = 2I_{15} & \Sigma\sigma_1^{15} + \Sigma s_1^{15} = 2\lambda_7\sqrt{\Delta}\, R_8
\end{array}
$$

where R_i is an invariant of order i.

§ 5. *The Discriminant of the Octavic.*

We have seen that $\Sigma\sigma_1^7 + \Sigma s_1^7 = 2\mu\sqrt{\Delta}$ when Δ denotes the discriminant of the octavic and μ is a numerical factor. To determine the value of μ, we select the particular octavic $x^8 - 1$, whose roots are the 8 eighth roots of unity, $1, a, a^2, \cdots, a^7$, where $a = \sqrt{i}$.

$$
\begin{aligned}
\sigma_1 = {} & (1-a)(a^2-a^3)(a^4-a^5)(a^6-a^7) \\
& + (1-a^2)(a-a^3)(a^6-a^4)(a^7-a^5) \\
& + (1-a^3)(a^2-a)(a^5-a^6)(a^7-a^4) \\
& + (1-a^4)(a^5-a)(a^2-a^6)(a^7-a^3) \\
& + (1-a^5)(a-a^4)(a^6-a^3)(a^7-a^2) \\
& + (1-a^6)(a^7-a)(a^3-a^5)(a^4-a^2) \\
& + (1-a^7)(a-a^6)(a^2-a^5)(a^3-a^4).
\end{aligned}
$$

Multiplying each factor by a is equivalent to multiplying each product by -1 and consequently sends σ into its negative. Multiplying each factor of σ

by a is also equivalent to operating on σ with the odd substitution $T \equiv (12345678)$. But $T(\sigma_1) = -s_4 = -\sigma_1$. Hence $\sigma_1 = s_4$.

Operating with T on all 15 σ's we get the following equalities:

$\sigma_1 = s_4$	$\sigma_4 = s_3$	$\sigma_7 = s_{15}$	$\sigma_{10} = s_{10}$	$\sigma_{13} = s_5$
$\sigma_2 = s_8$	$\sigma_5 = s_7$	$\sigma_8 = s_2$	$\sigma_{11} = s_{14}$	$\sigma_{14} = s_9$
$\sigma_3 = s_{12}$	$\sigma_6 = s_{11}$	$\sigma_9 = s_6$	$\sigma_{12} = s_1$	$\sigma_{15} = s_{13}$

These equalities among the 30 heptads show that for the octavic $x^8 - 1$ all the invariants of odd order are zero.

In addition to these equalities there are also equalities among the σ-heptads. These may be derived by operating on σ_i with the cyclic group of order 8: $1, T, T^2, \cdots, T^7$. Operating on σ_1 we get:

$$
\begin{aligned}
T\sigma_1 &= (12345678)\sigma_1 = -\sigma_1 = -s_4 \\
T^2\sigma_1 &= (1357)(2468)\sigma_1 = \sigma_1 \\
T^3\sigma_1 &= (14725836)\sigma_1 = -\sigma_1 = -s_4 \\
T^4\sigma_1 &= (15)(37)(26)(48)\sigma_1 = \sigma_1 \\
T^5\sigma_1 &= (16385274)\sigma_1 = -\sigma_1 = -s_4 \\
T^6\sigma_1 &= (1753)(2864)\sigma_1 = \sigma_1 \\
T^7\sigma_1 &= (18765432)\sigma_1 = -\sigma_1 = -s_4.
\end{aligned}
$$

Whence we see that σ_1 is invariant under the even substitutions of this group, and is only altered in sign by the odd substitutions.

Operating on σ_2 by the substitutions of this group we get:

$$\sigma_2 = \sigma_3 = \sigma_6 = \sigma_7 = s_8 = s_{11} = s_{12} = s_{15}.$$

Continuing this process we find that our 30 heptads reduce to 7 quantities.

$$
\begin{aligned}
&\sigma_1 = s_4, \\
&\sigma_2 = \sigma_2 = \sigma_6 = \sigma_7 = s_8 = s_{11} = s_{12} = s_{15}, \\
&\sigma_4 = \sigma_5 = s_3 = s_7, \\
&\sigma_8 = \sigma_9 = s_2 = s_6, \\
&\sigma_{10} = \sigma_{11} = \sigma_{14} = \sigma_{15} = s_9 = s_{10} = s_{13} = s_{15}, \\
&\sigma_{12} = s_1, \\
&\sigma_{13} = s_5.
\end{aligned}
$$

Hence we have:

$$2\mu\sqrt{\Delta} = 2(\sigma_1{}^7 + 4\sigma_2{}^7 + 2\sigma_4{}^7 + 2\sigma_8{}^7 + 4\sigma_{10}{}^7 + \sigma_{12}{}^7 + \sigma_{13}{}^7)$$

To find the values of σ_1, σ_2, σ_8, σ_{10}, we make use of the equations:

$$\begin{aligned}
\sigma_1 + s_4 &= 5(15\cdot 62\cdot 37\cdot 84) \\
\sigma_1 &= 5/2(15\cdot 62\cdot 37\cdot 84) = -40i \\
\sigma_2 + s_{12} &= 5(15\cdot 82\cdot 63\cdot 47) \\
\sigma_2 &= 5/2(15\cdot 82\cdot 63\cdot 47) = -10i(1+\sqrt{2}) \\
\sigma_8 + s_2 &= 5(17\cdot 53\cdot 26\cdot 48) \\
\sigma_8 &= 5/2(17\cdot 53\cdot 26\cdot 48) = 20_i \\
\sigma_{10} + s_{10} &= 5(15\cdot 87\cdot 32\cdot 64) \\
\sigma_{10} &= 5/2(15\cdot 87\cdot 32\cdot 64) = -10i(1-\sqrt{2}).
\end{aligned}$$

We can not obtain σ_4, σ_{12} and σ_{13} in this way, because they are not equal to any of their correlates. We obtain σ_4 by straight substitution, getting

$$\begin{aligned}
&\sigma_4 = 30i \\
&\sigma_{12} = 10i\,(1+2\sqrt{2}) \\
&\sigma_{13} = 10i\,(1-2\sqrt{2}). \\
&\mu\sqrt{\Delta} = 10^7 i\,\{4^7 + 4[(1+\sqrt{2})^7 + (1-\sqrt{2})^7] \\
&\qquad\qquad - [(1+2\sqrt{2})^7 + (1-2\sqrt{2})^7] - 2\cdot 3^7 - 2^8\} \\
&\qquad = 1680\, i\, 10^7.
\end{aligned}$$

The best way to get the product of the differences of the roots $\overset{28}{\Pi}$ is to note that

$$(\sigma + s_1)(\sigma_1 + s_2)(\sigma_1 + s_3)(\sigma_1 + s_4)(\sigma_1 + s_5)\,(\sigma_1 + s_6)(\sigma_1 + s_7) = 5^7\overset{28}{\Pi}$$

$$\begin{aligned}
&\sigma_1 + s_1 = -10i\,(3-2\sqrt{i}) \\
&\sigma_1 + s_2 = -20i \\
&\sigma_1 + s_3 = -10i \\
&\sigma_1 + s_4 = -80i \\
&\sigma_1 + s_5 = -10i\,(3+2\sqrt{2}) \\
&\sigma_1 + s_6 = -20i \\
&\sigma_1 + s_7 = -10i \\
&5^7\overset{28}{\Pi} = i\,10^7(3-2\sqrt{2})(3+2\sqrt{2})\,2\cdot 2\cdot 8 = 2^{12}\,5^7\,i \\
&\overset{\infty}{\Pi} = 2^{12} i \\
&\sqrt{\Delta} = \pm\overset{28}{\Pi}/8^4 = \pm\overset{28}{\Pi}/2^{12} = \pm i \\
&\mu\sqrt{\Delta} = \pm\mu i = 1680 i\cdot 10^7 \\
&\mu = \pm 1680\cdot 10^7 = \pm 2^{11}\cdot 3\cdot 5^8\cdot 7 \\
&\Sigma\sigma_1^7 + \Sigma s_1^7 = 2\mu\sqrt{\Delta} = \pm 2^{12}\cdot 3\cdot 5^8\cdot 7\sqrt{\Delta}.
\end{aligned}$$

§ 6. *The Equation of the Fifteen σ-Heptads.*

Let us write the equation whose roots are the 15 σ-heptads in the form

$$x^{15} + \kappa_2 x^{13} + \kappa_3 x^{12} + \kappa_4 x^{11} + \cdots + \kappa_{15} = 0.$$

Making use of Salmon's Table,* giving the sums of the products of the roots of an equation in terms of the powers of the roots, and noting that $\Sigma\,\sigma_i{}^7 = I_7 + \mu\sqrt{\Delta}$ we obtain

$$\begin{aligned}
\kappa_2 &= \Sigma\,\sigma_1\sigma_2 = -I_2/2 \\
\kappa_3 &= -\Sigma\,\sigma_1\sigma_2\sigma_3 = -I_3/3 \\
\kappa_4 &= \Sigma\,\sigma_1\sigma_2\sigma_3\sigma_4 = \tfrac{1}{8}(I_2{}^2 - 2I_4) \\
\kappa_5 &= -\Sigma\,\sigma_1\cdots\sigma_5 = -I_5/5 + I_2I_3/6 \\
\kappa_6 &= \Sigma\,\sigma_1\sigma_2\cdots\sigma_6 = -I_2{}^3/48 + I_2I_4/8 + I_3{}^2/18 - I_6/6 \\
\kappa_7 &= -\Sigma\,\sigma_1\sigma_2\cdots\sigma_7 = I_2I_5/10 - I_2{}^2I_3/24 + I_3I_4/12 \\
&\qquad - I_7/7 - \mu/7\,\sqrt{\Delta} \\
\kappa_8 &= \Sigma\,\sigma_1\cdots\sigma_8 = I_2{}^4/384 - I_2{}^2I_4/32 - I_2I_3{}^2/36 + I_2I_6/12 \\
&\qquad + I_3I_5/15 + I_4{}^2/32 - I_8/8 \\
\kappa_9 &= -\Sigma\,\sigma_1\sigma_2\cdots\sigma_9 = I_4I_5/20 - I_2{}^2I_5/40 + I_2{}^3I_3/144 - I_2I_3I_4/24 \\
&\qquad - I_3{}^2/162 + I_2I_7/14 + I_3I_6/18 - I_9/9 \\
&\qquad - (\lambda_1/9 - \mu/14)I_2\sqrt{\Delta} \\
\kappa_{10} &= \Sigma\,\sigma_1\sigma_2\cdots\sigma_{10} = -I_2{}^5/3840 + I_2{}^3\lambda_4/192 + I_2{}^2I_3{}^2/144 \\
&\qquad - I_2{}^2I_6/48 - I_2I_3I_5/30 - I_2I_4{}^2/64 + I_2I_8/16 \\
&\qquad + I_3I_7/21 - I_3{}^2I_4/72 + I_4I_6/24 + I_5{}^2/50 \\
&\qquad - I_{10}/10 + (\mu/21 - \lambda_2/10)\,I_3\sqrt{\Delta} \\
\kappa_{11} &= -\Sigma\,\sigma_1\cdots\sigma_{11} = I_2{}^3I_5/240 - I_2I_4I_5/40 - I_3{}^2I_5/90 - I_2{}^4I_3/1152 \\
&\qquad + I^2{}_2I_3I_4/96 - I_2{}^2I_7/56 + I_2I_3{}^3/324 - I_2I_3I_6/36 \\
&\qquad + I_2I_9/18 + I_5I_6/30 - I_3I_4{}^2/96 + I_3I_8/24 \\
&\qquad + I_4I_7/28 - I_{11}/11 - ({}^{\mu}/_{56} - {}^{\lambda_1}/_{18})I_2{}^2\sqrt{\Delta} \\
&\qquad - (\lambda_3R_4/11 - \mu I_4/28)\sqrt{\Delta}
\end{aligned}$$

§ 7. *The Equation of the Thirty Heptads: A Resolvent Equation of Degree Thirty of the Octavic.*

The equation whose roots are $-s_1, -s_2, \cdots, -s_{15}$ may be obtained from the equation of the 15 σ-heptads by changing the sign of $\sqrt{\Delta}$. Denot-

* Salmon, *Higher Algebra*, p. 307.

ing by $F(\sigma)$ and $F(-s)$ the equations of the σ- and $-s$-heptads respectively we have:

$$F(\sigma) = x^{15} - I_2/2\, x^{13} - I_3/3\, x^{12} + \cdots + \sqrt{\Delta}(-\mu/7\, x^8 + \cdots) = 0$$
$$F(-s) = x^{15} - I_2/2\, x^{13} - I_3/3\, x^{12} + \cdots - \sqrt{\Delta}(-\mu/7\, x^8 + \cdots) = 0.$$

Multiplying these two equations we get an equation of degree 30 with roots $\sigma_1, \sigma_2, \cdots, \sigma_{13}, -s_1, \cdots, -s_{15}$. Denoting this product by F_{30} we have:

$$F_{30} = (x^{15} - I_2/2\, x^{13} - \cdots)^2 - \Delta(-\mu/7\, x^8 + \cdots)^2 = 0.$$

Noether* defining a set of 30 functions Σ, corresponding to the heptads considered here, shows that the equation which they satisfy may be used as a resolvent equation for the octavic.

The sums of the nth powers of the roots of F_{30} are the invariants $2I_n$. Consequently we may write F_{30} without using Δ explicitly.

$$\begin{aligned} F_{30} = x^{30} - I_2 x^{28} - 2I_3/3\, x^{27} - (I_2^2/2 - I_4/2)x^{26} + 2(I_2 I_3/3 - I_5/5)x^{25} \\ + (I_2 I_4/2 - I_2^3/6 + 2I_3^2/9 - I_6/3)x^{24} \\ + (2/5\, I_2 I_5 - 2I_7/7 + I_3 I_4/3 - I_2^2 I_3/3)x^{23} \\ + (I_4^2/8 - I_8/4 + I_2 I_6/3 + I_2^4/24 - I_2^2 I_4/4 + 4I_3 I_5/15 \\ - 2I_2 I_3^2/9)x^{22} \\ + (2/7\, I_2 I_7 - 2I_9/9 + 2I_3 I_6/9 - I_2^2 I_5/5 - I_2 I_3 I_4/3 \\ + I_4 I_5/5 - 4/81\, I_3^3 + I_2^3 I_3/9)x^{21} \\ + (I_2^3 I_4/12 - I_2^5/120 + I_2^2 I_3^2/9 - I_2^2 I_6/6 - I_2 I_4^2/8 \\ + I_2 I_8/4 + 4/21\, I_3 I_7 + I_4 I_6/6 - I_3^2 I_4 + 2I_5^2/25 - I_{10}/5)x^{20} \\ + \cdots. \end{aligned}$$

§ 8. *Significance of Equalities among the Heptads.*

If Δ, the discriminant of the octavic, is zero, each σ-heptad is equal to the negative of one of its seven correlates, and F_{30} is a perfect square. Conversely if any heptad is equal to the negative of one of its correlates, every heptad is equal to the negative of one of its correlates, and the discriminant of the octavic is zero since

$$\Delta = \lambda(\sigma_1 + s_1)^2(\sigma_1 + s_2)^2(\sigma_1 + s_3)^2(\sigma_1 + s_4)^2(\sigma_1 + s_5)^2 (\sigma_1 + s_6)^2(\sigma_1 + s_7)^2$$

an equation which has 30 equivalent forms.

* "Ueber die Gleichungen achten Grades," *Mathematische Annalen,* XV (1879).

THEOREM: *If the discriminant of the octavic is not zero, and any two of the σ-heptads or of the s-heptads are equal, the roots of the octavic divide into two sets of four points which have the same cross ratio.*

To prove this we note that two σ-heptads determine a triad and any triad picks out a division of the eight points into two sets of four. Let the two equal heptads be σ_1 and σ_2. These pick out the triad $\sigma_1\sigma_2\sigma_3$ which gives the division (1234) (5678).

Now $\sigma_1 + s_1 = 5 \cdot 12 \cdot 34 \cdot 56 \cdot 78$, $\sigma_2 + s_1 = 5 \cdot 14 \cdot 23 \cdot 57 \cdot 86$.

$\sigma_1 = \sigma_2$ gives $12 \cdot 34 \cdot 56 \cdot 78 = 14.23.57.86$,

whence $$\frac{12 \cdot 34}{14 \cdot 23} = \frac{57 \cdot 86}{56 \cdot 78} \quad \text{or } \{13 \cdot 24\} = \{58 \cdot 76\}.$$

Hence we have proved that if two σ-heptads are equal, the octavic divides into two projective sets of four points. The argument is the same for two equal s-heptads. Conversely if the octavic divides into two projective sets of four points, there are two equal σ-heptads or two equal s-heptads.

The invariant, whose vanishing implies the division of the roots of the octavic into two projective sets is then:

$$(\sigma_1 - \sigma_2)^2(\sigma_1 - \sigma_3)^2 \cdot \cdot \cdot, \qquad (s_1 - s_2)^2(s_1 - s_3)^2 \cdot \cdot \cdot,$$

which is of degree 420 in the heptads and consequently of degree 420 in the roots of the octavic. This invariant which we may call $C^0{}_{420}$ is the product of the discriminant of $F(\sigma)$ and the discriminant of $F(-s)$.

THEOREM: *If the three heptads in any triad are equal* $(\Delta \neq 0)$ *the roots of the octavic fall into two self apolar sets.*

Let $\sigma_1 = \sigma_2 = \sigma_3$.

$$\sigma_1 = \sigma_2 \text{ gives us } \{13 \cdot 24\} = \{58 \cdot 76\}$$
$$\sigma_1 = \sigma_3 \text{ gives us } \{14 \cdot 23\} = \{57 \cdot 86\}.$$

Let $\{13 \cdot 24\} = \{58 \cdot 76\} = \lambda$

then $\{14 \cdot 23\} = \lambda/(\lambda - 1)$, $\{57 \cdot 86\} = 1 - \lambda$,

whence $\lambda/(\lambda - 1) = 1 - \lambda$, or $\lambda^2 - \lambda + 1 = 0$.

$\therefore \lambda = -\omega$ or $-\omega^2$ which proves the theorem.

THEOREM: *If two heptads of a σ-triad and two heptads of the corresponding s-triad are equal* $(\Delta \neq 0)$, *the roots of the octavic form two harmonic sets.*

Conversely if the roots of the octavic form two harmonic sets two of the σ-heptads and two of the corresponding s-heptads are equal.

By corresponding triads we mean the two triads whose six heptads add up to zero, e. g. $\sigma_1\sigma_2\sigma_3$, $s_1s_2s_3$, are corresponding triads.

Let us suppose $\sigma_1 = \sigma_2$ and $s_1 = s_2$.

$$\sigma_1 + s_2 = 5\cdot13\cdot24\cdot75\cdot86, \quad \sigma_2 + s_1 = 5\cdot14\cdot23\cdot57\cdot86$$

$$\frac{\sigma_1 + s_2}{\sigma_2 + s_1} = -\frac{13\cdot24}{14\cdot23} = 1.$$

$$\therefore \{12\cdot34\} = -1 = \{57\cdot86\}$$

which proves the theorem.

In the preceding discussion we have assumed that there are no equalities among the roots of the octavic. If the octavic has only one double root the theorems still hold; but the proofs break down in case of two double roots or of a triple root.

§ 9. *Derivation of the Necessary Invariant Conditions that an Octavic Have a Triple Root.*

Maisano,* using the ordinary system of invariants, has shown that, if an octavic have a triple root, five independent invariant relations must vanish. These are of orders 6, 7, 8, 9, 10. In the system of invariants, set forth in this paper, the necessary conditions for a triple root are readily derived. Let us assume that three of the roots of the octavic, say 6, 7, 8 coincide. Now the roots 6, 7, 8 pick out 5 σ-triads and 5 corresponding s-triads, corresponding to the divisions:

$$(1678), \quad (2678) = (1345), \quad (3678) = (1245)$$
$$(4678) = (1235), \quad (5678) = (1234).$$

Since 6, 7, 8 are all equal the three heptads in the triad corresponding to the division (1678) must be equal, since they are obtainable, one from the other by permutations on the roots 6, 7, 8. Likewise the three heptads in the corresponding s-triad are all equal, and equal to the negative of the heptads

* "L'Hessiano della sestica binaria e il discriminante della forma dell' ottavo ordine," *Rendiconti del Cercolo Matematico di Palermo*, 4 (1890).

in the σ-triad. Similarly the heptads on the divisions (2678) = (1352) are all equal, etc. Hence we have the following equalities:

$$\sigma_1 = \sigma_2 = \sigma_3 = - s_1 = - s_2 = - s_3$$
$$\sigma_4 = \sigma_{11} = \sigma_{15} = - s_5 = - s_{11} = - s_{14}$$
$$\sigma_5 = \sigma_9 = \sigma_{12} = - s_4 = - s_8 = - s_{12}$$
$$\sigma_6 = \sigma_8 = \sigma_{14} = - s_6 = - s_9 = - s_{15}$$
$$\sigma_7 = \sigma_{10} = \sigma_{13} = - s_7 = - s_{10} = - s_{13}.$$

Thus we have the fact that, if the octavic has a triple root, the equations $F(\sigma)$ and $F(-s)$ become perfect cubes.

$$\begin{aligned} F(\sigma) &= x^{15} + \kappa_2 x^{13} + \cdots \kappa_{15} \\ &= (x^5 + A_2x^3 + A_3x^2 + A_4x + A_5).^3. \end{aligned}$$

Whence we have:

$$\kappa_2 = 3A_2, \quad \kappa_3 = 3A_3, \quad \kappa_4 = 3A_2^2 + 3A_4, \; \kappa_5 = 3A_5 + 6A_2A_3,$$
$$\kappa_6 = A_2^3 + 3A_3^2 + 6A_2A_4, \quad \kappa_7 = 6A_3A_4 + 6A_2A_5 + 3A_2^2A_3,$$
$$\kappa_8 = 6A_3A_5 + 3A_2^2A_4 + 3A_3^2A_2 + 3A_4^2,$$
$$\kappa_9 = A_3^3 + 3A_2^2A_5 + 6A_2A_3A_4 + 6A_4A_5,$$
$$\kappa_{10} = 3A_5^2 + 6A_2A_3A_5 + 3A_2A_4^2 + 3A_3^2A_4,$$
$$\kappa_{11} = 3A_3^2A_5 + 3A_4^2A_3 + 6A_2A_4A_5,$$
$$\kappa_{12} = 6A_3A_4A_5 + A_4^3 + 3A_5^2A_2, \quad \kappa_{13} = 3A_5^2A_3 + 3A_4^2A_5,$$
$$\kappa_{14} = 3A_5^2A_4, \quad \kappa_{15} = A_5^3.$$

From these equations we derive the following five independent relations:

$$R_6 = 27\kappa_6 + 5\kappa_2^3 - 9\kappa_3^2 - 18\kappa_2\kappa_4 = 0.$$
$$R_7 = -9\kappa_7 + 6\kappa_3\kappa_4 + 6\kappa_2\kappa_5 - 5\kappa_2^2\kappa_3 = 0.$$
$$R_8 = 9\kappa_8 - 6\kappa_3\kappa_5 + 3\kappa_2\kappa_4^2 + \kappa_2^2\kappa_4 - 3\kappa_4^2 = 0.$$
$$R_9 = 27\kappa^9 - \kappa_3^3 + 3\kappa_2^2\kappa_5 - 18\kappa_4\kappa_5 + 6\kappa_2\kappa_3\kappa_4 = 0.$$
$$\begin{aligned} R_{10} = 81\kappa_{10} &- 27\kappa_5^2 - \kappa_2^5 - 9\kappa_2\kappa_4^2 + 6\kappa_2^3\kappa_4 \\ &- 9\kappa_3^2\kappa_4 + 3\kappa_2^2\kappa_3^2 + 18\kappa_2\kappa_3\kappa_5 = 0. \end{aligned}$$

A number of additional relations exist, but they are simply combinations of these five.

Substituting for the coefficients $\kappa_2, \kappa_3, \cdots$, we get

$$R_6 = 8I_3^2 + 18I_2I_4 - I_2^3 - 72I_6.$$
$$R_7 = 1080I_7 - 252I_2I_5 + 35I_2^2I_3 - 210I_3I_4.$$
$$R_8 = 2160I_8 - 180I_4^2 - 384I_3I_5 - 1440I_2I_6.$$

$$R_9 = 45360I_9 - 22680I_3I_6 - 29160I_2I_7 - 6804I_4I_5.$$
$$+ 5670I_2^2I_5 + 9450I_2I_3I_4 + 1960I_3^3 - 945I_2^3I_3.$$
$$R_{10} = 1088640I_{10} + 1785I_2^5 - 36400I_2^2I_3^2 - 680400I_2I_8$$
$$- 518400I_3I_7 - 453600I_4I_6 - 72576I_5^2 + 201600I_2I_3I_5$$
$$+ 226800I_2^2I_6 + 132300I_2I_4^2 - 44100I_2^3I_4 + 117600I_3^2I_4.$$

Following Maisano, we may write the discriminant of the octavic in the form

$$\Delta = (\lambda_1I_8 + \lambda_2I_2I_6 + \lambda_3I_4^2 + \cdots \lambda_7I_2^4)R_6$$
$$+ (\mu_1I_7 + \mu_2I_3I_4 + \cdots)R_7 + (\nu_1I_6 + \cdots)R_8$$
$$+ (\delta_1I_5 + \delta_2I_2I_3)R_9 + (\epsilon_1I_4 + \epsilon_2I_2^2)R_{10},$$

which reduces the problem of determining the discriminant of the octavic in terms of the fundamental invariants to the determination of 19 constants instead of 31. Maisano accomplishes the determination of these constants, using the ordinary scheme of invariants. The work involves a considerable amount of algebra and we shall not attempt it in this paper. The determination of Δ in terms of the fundamental invariants, at the same time gives us I_{14} for from

$$\Sigma\,\sigma_1^7 - \Sigma\,s_1^7 = 2I_7 \text{ and } \Sigma\,\sigma_1^7 + \Sigma\,s_1^7 = 2\mu\sqrt{\Delta} \text{ we derive}$$
$$I_{14} = I_7^2 + \mu^2\Delta.$$

§ 9. *Determination of I_{11} as a rational function of the fundamental invariants.*

The invariant I_{11} is not a fundamental invariant. There exists, therefore, a relationship of degree 11, an $\overline{R}_{11}$ defining I_{11} rationally in terms of the fundamental system. In the case of the octavic with a triple root this general $\overline{R}_{11}$ reduces to

$$R_{11} = 81\kappa_{11} - 9\kappa_3^2\kappa_5 + 6\kappa_2\kappa_3^3 - 9\kappa_4^2\kappa_3 - 5\kappa_2^4\kappa_3 + 18\kappa_2^2\kappa_3\kappa_4$$
$$- 18\kappa_2\kappa_4\kappa_5 + 6\kappa_2^3\kappa_5 = 0.$$

Hence in general we must have

$$\overline{R}_{11} = R_{11} + \lambda_1\kappa_2R_9 + \lambda_2\kappa_3R_8 + \lambda_3\kappa_4R_7 + \lambda_4\kappa_2^2R_7$$
$$+ \lambda_5\kappa_5R_6 + \lambda_3\kappa_2\kappa_3R_6 = 0,$$

where $\lambda_1 \cdots \lambda_6$ are numerical factors to be determined from particular cases. For this purpose we shall consider the octavic with two pairs of equal roots. Let us suppose that the pairs of roots (12) and (34) are equal.

The transformations (12), (34), and (12)(34) must then leave every

heptad unaltered. From this we find our heptad system pairing off as follows:

$$\sigma_1,\ \sigma_2,\ \sigma_3 \text{ distinct; } \quad s_1,\ s_2,\ s_3 \text{ distinct;}$$

$$\sigma_1 = -s_1;\ \sigma_2 = -s_2;\ \sigma_3 = -s_3;\ \sigma_4 = \sigma_5 = -s_4 = -s_5;$$

$$\sigma_6 = \sigma_7 = -s_6 = -s_7;\ \sigma_8 = \sigma_{10} = -s_{13} = -s_{15};$$

$$\sigma_9 = \sigma_{11} = -s_{12} = -s_{14};\ \sigma_{12} = \sigma_{15} = -s_8 = -s_{11};$$

$$\sigma_{13} = \sigma_{14} = -s_9 = -s_{10}.$$

Thus when the octavic has two double roots, the fifteenic $F(\sigma)$ has six double roots. Let us now consider the octavic with roots $0,\ 0,\ \infty,\ \infty,\ a',\ b',\ c',\ d'$, and let $x^4 - \alpha_1 x^3 + \alpha_2 x^2 - \alpha_3 x + \alpha_4 = 0$ denote the equation whose roots are $a,\ b,\ c,\ d$ where $a = \sqrt{5}\,a'$, etc. Our 15 σ-heptads then take the values

$$\begin{array}{lll} ab + cd - \alpha_2, & ac + bd - \alpha_2, & ad + bc - \alpha_2, \\ & ab - \alpha_2\ ]^2, & bc - \alpha_2\ ]^2, \\ & ac - \alpha_2\ ]^2, & bd - \alpha_2\ ]^2, \\ & ad - \alpha_2\ ]^2, & cd - \alpha_2\ ]^2, \end{array}$$

where the notation $]^2$ means that there are two heptads with the enclosed values. Making the substitutions:

$$a + b = u; \quad c + d = v; \quad ab = x_1; \quad cd = y_1,$$

we may write:

$$u + v = \alpha_1; \quad uv + x_1 + y_1 = \alpha_2; \quad uy_1 + vx_1 = \alpha_3; \quad x_1 y_1 = \alpha_4.$$

$$x_1\alpha_1 = x_1 u + x_1 v; \quad x_1\alpha_1 - \alpha_3 = (x_1 - y_1)u;$$

$$y_1(\alpha_1 - \alpha_3) = (y_1 - x_1)v;$$

whence we obtain the equation:

$$(x_1\alpha_1 - \alpha_3)(y_1\alpha_1 - \alpha_3) = -(x_1 - y_1)^2(\alpha_2 - x_1 - y_1)$$

Putting $x = x_1 + y_1$ and $\alpha_2 = 0$ we get the equation with roots $ab + cd$, etc.

$$x^3 + (\alpha_1\alpha_3 - 4\alpha_4)x - (\alpha_3^2 + \alpha_1^2\alpha_4) = 0. \tag{1}$$

Denote the roots of the equation by $\lambda_1,\ \lambda_2,\ \lambda_3$.

$$(x - ab)(x - cd) = x^2 - \lambda_1 x + \alpha_4 = 0$$

$$(x - ac)(x - bd) = x^2 - \lambda_2 x + \alpha_4 = 0$$

$$(x - ad)(x - bc) = x^2 - \lambda_3 x + \alpha_4 = 0$$

Multiplying these three quadratics, we obtain the sextic whose roots are *ab, ac, ad, bc, bd, cd.*

$$x^6 + (a_1a_3 - a_4)x^4 - (a_3^2 + a_1^2a_4)x^3 + (3a_4^2 + a_1a_3a_4 - 4a_4^2)x^2 + a_4^3.$$

Squaring this sextic and multiplying by cubic (1) we obtain the equation of our 15 σ-heptads:

$$x^{15} + \kappa_2x^{13} + \kappa_3x^{12} + \cdots \kappa_{15} = 0$$

where

$$\kappa_2 = 3a_1a_3 - 6a_4, \quad \kappa_3 = -(3a_3^2 + 3a_1^2a_4),$$
$$\kappa_4 = 3a_1^2a_3^2 + 7a_4^2 - 10a_1a_3a_4,$$
$$\kappa_5 = -(6a_1a_3^3 - 12a_3^2a_4 + 6a_1^3a_3a_4 - 12a_1^2a_4^2)$$
$$\kappa_6 = 3a_3^4 + 4a_1^2a_3^2a_4 + 3a_1^4a_4^2 - 5a_1a_3a_4^2 + 8a_4^3 + a_1^3a_3^3$$
$$\kappa_7 = -(5a_3^2a_4^2 + 5a_1^2a_4^3 + 3a_1^2a_3^4 + 3a_1^4a_3^2a_4 - 8a_1a_3^3a_4 - 8a_1^3a_3a_4^2)$$
$$\kappa_8 = 3a_1a_3^5 - 23a_1^2a_3^2a_4^2 - 17a_4^4 + 8a_1^3a_3^3a_4 + 3a_1^5a_3a_4^2 + 20a_1a_3a_4^3 - 6a_1^4a_4^3 - 6a_3^4a_4$$
$$\kappa_9 = -(14a_3^2a_4^3 + 14a_1^2a_4^4 + 7a_1^2a_3^4a_4 + 7a_1^4a_3^2a_4^2 - 14a_1a_3^3a_4^2 - 14a_1^3a_3a_4^3 + a_3^6 + a_1^6a_4^3)$$
$$\kappa_{10} = a_1a_3a_4^4 + 2a_4^5 + 5a_1^3a_3^3a_4^2 - 8a_1^2a_3^2a_4^3 + 2a_1a_3^5a_4 - 2a_3^4a_4^2 + 2a_1^5a_3a_4^3 - 2a_1^4a_4^4$$
$$\kappa_{11} = -2a_1^3a_3a_4^4 - 2a_1a_3^3a_4^3 + 9a_1^2a_4^5 + 9a_3^2a_4^4 - a_1^2a_3^4a_4^2 - a_1^4a_3^2a_4^3.$$

For the determination of I_{11} we shall need six cases:

Case 1. Let $a_1 = a_3 = a_4 = 1$.

then

$\kappa_2 = -3$	$\kappa_6 = 14$	$\kappa_{10} = 0$
$\kappa_3 = -6$	$\kappa_7 = 0$	$\kappa_{11} = 12$
$\kappa_4 = 0$	$\kappa_8 = -18$	
$\kappa_5 = 12$	$\kappa_9 = -16$	

$R_6 = -81$	$R_9 = 108$
$R_7 = 54$	$R_{11} = 1458.$
$R_8 = -54$	

which gives us the equation

$$-324\lambda_1 + 324\lambda_2 + 486\lambda_4 - 972\lambda_5 - 1458\lambda_6 + 1458 = 0,$$

which simplifies to:

$$2\lambda_1 - 2\lambda_2 - 3\lambda_4 + 6\lambda_5 + 9\lambda_6 = 9. \tag{1}$$

Case 2. $a_1 = a_4 = 1$; $a_3 = 0$.

$\kappa_2 = -6$	$\kappa_5 = 12$	$\kappa_8 = -23$
$\kappa_3 = -3$	$\kappa_6 = 11$	$\kappa_9 = -15$
$\kappa_4 = 7$	$\kappa_7 = -5$	$\kappa_{10} = 0$
		$\kappa_{11} = 9$
$R_6 = -108$	$R_8 = -48$	$R_{11} = 1404.$
$R_7 = 27$	$R_9 = 162$	

These values give us the equation

$$-972\lambda_1 + 144\lambda_2 + 189\lambda_3 + 972\lambda_4 - 1296\lambda_5 - 1944\lambda_6 = -1404,$$

which reduces to

$$108\lambda_1 - 16\lambda_2 - 21\lambda_3 - 108\lambda_4 + 144\lambda_5 + 216\lambda_6 = 156. \tag{2}$$

Case 3. $a_1 = a_4 = 1$; $a_3 = 2$.

$\kappa_2 = 0$	$\kappa_6 = 73$	$\kappa_{10} = 46$
$\kappa_3 = -15$	$\kappa_7 = -5$	$\kappa_{11} = 5$
$\kappa_4 = -1$	$\kappa_8 = -5$	
$\kappa_5 = 0$	$\kappa_9 = -135$	
$R_6 = -54$	$R_8 = -48$	$R_{11} = 540.$
$R_7 = 135$	$R_9 = -270$	

These values give us the equation:

$$720\lambda_2 - 135\lambda_3 = -540 \quad \text{or}$$
$$16\lambda_2 - 3\lambda_3 = -12. \tag{3}$$

From the case of the octavic with two double roots and $a_2 = 0$, we were unable to obtain more than three independent equations. The remaining three equations we obtain under the assumption $a_2 \neq 0$.

We have seen, that in the case of a double root at zero and infinity, our 15 heptads have the values:

$$\begin{array}{lll} 5(ab + cd) - a_2, & 5ab - a_2\,]^2, & 5bc - a_2\,]^2, \\ 5(ac + bd) - a_2, & 5ac - a_2\,]^2, & 5bd - a_2\,]^2, \\ 5(ad + bc) - a_2, & 5ad - a_2\,]^2, & 5cd - a_2\,]^2, \end{array}$$

where a, b, c, d are the remaining four roots and $a_2 = \Sigma\, ab$.

Let us give a, b, c, d the particular values: $a = 1/\sqrt{5}$, $b = -1/\sqrt{5}$, $c = 2/\sqrt{5}$, $d = -2/\sqrt{5}$ then $a_2 = -1$.

Our 15 σ's then take the values:

$$-4,\ 5,\ -3\,]^3,\ 3\,]^4,\ -1\,]^4,\ 0\,]^2,$$

so that

$$\begin{aligned}F(\sigma) &= x^2(x+1)^4(x-3)^4(x+3)^3(x+4)(x-5)\\ &= x^2(x^{13} - 54x^{11} - 28x^{10} + 1095x^9 + 1044x^8 - 10388x^7\\ &\quad - 14520x^6 + 43551x^5 + 88776x^4 - 35478x^3\\ &\quad - 195372x^2 - 162567x - 43740).\end{aligned}$$

$\kappa_2 = -54$	$\kappa_8 = 43551$	$\kappa_4 R_7 = 18290880$
$\kappa_3 = -28$	$\kappa_9 = 88776$	$\kappa_2{}^2 R_7 = 48708864$
$\kappa_4 = 1095$	$\kappa_{10} = -35478$	$\kappa_5 R_6 = -10974528$
$\kappa_5 = 1044$	$\kappa_{11} = -195372.$	$\kappa_2\kappa_3 R_6 = -15894144$
$\kappa_6 = -10388$	$\kappa_2 R_9 = -49054464$	
$\kappa_7 = -14520$	$\kappa_3 R_8 = -1016064$	

$R_6 = -10512$	$R_9 = 908416$
$R_7 = 16704$	$R_{11} = -7962624$
$R_8 = 36288$	

These values give us the equation:

$$\begin{aligned}28388\lambda_1 + 588\lambda_2 - 10585\lambda_3 - 28188\lambda_4 + 6351\lambda_5\\ + 9198\lambda_6 = -4608.\end{aligned} \tag{4}$$

Case 5.

$$a = 1/\sqrt{5},\quad b = i/\sqrt{5},\quad c = \sqrt{5} - 1/\sqrt{5},\quad d = -i/\sqrt{5},\quad a_2 = 1.$$

The 15 σ's are

$$\begin{aligned}&3i - 1\,],\ -(3i+1)\,],\ 4\,],\ (i-1)\,]^2,\ -(i+1)\,]^2,\ (4i-1)\,]^2,\\ &\quad -(4i+1)\,]^2,\ 3\,]^2,\ 0\,]^2.\end{aligned}$$

$$\begin{aligned}F(\sigma) &= x^2(x^3 - 2x^2 + 2x - 40)(x^2 + 2x + 2)^2(x^3 - x^2 + 11x - 51)^2\\ &= x^2(x^{13} + 21x^{11} - 122x^{10} - 55x^9 - 1844x^8 + 2487x^7\\ &\quad + 6670x^6 + 47126x^5 + 36424x^4 - 105428x^3\\ &\quad - 497128x^2 - 631992x - 416160) = 0.\end{aligned}$$

$\kappa_2 = 21$	$\kappa_7 = 6670$	$R_7 = 16896$
$\kappa_3 = -122$	$\kappa_8 = 47126$	$R_8 = -21312$
$\kappa_4 = -55$	$\kappa_9 = 36424$	$R_9 = -620416$
$\kappa_5 = -1844$	$\kappa_{11} = -497128$	$R_{11} = 12495024.$
$\kappa_6 = 2487$	$R_6 = 288$	

From these we get the equation:

$$271432\lambda_1 - 54168\lambda_2 + 19360\lambda_3 - 155232\lambda_4 + 11064\lambda_5 + 15372\lambda_6 = 260313. \qquad (5)$$

Case 6. Let a, b, c, d take such values that

$$5(ab + cd) - \alpha_2 = 0 \qquad (1)$$
$$5(ac + bd) - \alpha_2 = 0 \qquad (2)$$
$$5ab - \alpha_2 = 1 \qquad (3)$$
$$5ac - \alpha_2 = 1 \qquad (4)$$
$$\alpha_2 = -2. \qquad (5)$$

Our heptad system is then determined. From (1) and (2) we get

$$5(ad + bc) - \alpha_2 = 2\alpha_2 = -4.$$

From (3), (4), (5) we get

$$5ab = -1, \qquad 5ac = -1, \qquad 5cd = -1,$$

whence $b = c, \quad a = d, \quad 5ac = 5bd = -1,$

$$5bd - \alpha_2 = 1, \qquad 5cd - \alpha_2 = 1.$$

It remains to determine the values of $5ad - \alpha_2$ and $5bc - \alpha_2$. We shall determine the quadratic of which these heptads are the roots

$$x^2 + \beta x + \gamma = 0$$
$$-\beta = 5(ad + bc) - 2\alpha_2 = -2$$
$$\gamma = 25abcd - 5\alpha_2(ad + bc) + \alpha_2^2$$
$$= 25a^2b^2 - 12 + 4 = 1 - 12 + 4 = -7.$$

Hence the required quadratic is

$$x^2 + 2x - 7 = 0.$$

Our 15 heptads have the values:

$$0\,]^2, \quad 1\,]^8, \quad -4\,], \quad r_1\,]^2, \quad r_2\,]^2$$

where r_1 and r_2 are the roots of $x^2 + 2x - 7 = 0$.

$$F(\sigma) = x^2(x-1)^8(x+4)(x^2+2x-7)^2 = 0$$
$$= x^2(x^{13} - 30x^{11} + 52x^{10} + 271x^9 - 1036x^8 + 476x^7 + 4136x^6 - 11473x^5 + 15304x^4 - 12190x^3 + 5924x^2 - 1631x + 196).$$

$\kappa_2 = -30$ $\kappa_6 = 476$ $\kappa_{10} = -12190$
$\kappa_3 = 52$ $\kappa_7 = 4136$ $\kappa_{11} = 5924$
$\kappa_4 = 271$ $\kappa_8 = -11473$
$\kappa_5 = -1036$ $\kappa_9 = 15304$

$R_6 = -144$ $\kappa_2 R_9 = 226560$
$R_7 = -192$ $\kappa_3 R_8 = 9984$
$R_8 = 192$ $\kappa_4 R_7 = -52032$
$R_9 = -7552$ $\kappa_2^2 R_7 = -172800$
$R_{11} = -73728$ $\kappa_5 R_6 = 149184$
$\kappa_2\kappa_3 R_6 = 224640.$

From these we get the equation:

$$1180\lambda_1 + 52\lambda_2 - 271\lambda_3 - 900\lambda_4 + 777\lambda_5 + 1170\lambda_6 = 384. \tag{6}$$

We have now obtained the following six equations to determine the six constants $\lambda_1, \lambda, \cdot \cdot \cdot, \lambda_6$.

(1) $2\lambda_1 - 2\lambda_2 - 3\lambda_4 + 6\lambda_5 + 9\lambda_6 = 9$ (1)

(2) $108\lambda_1 - 16\lambda_2 - 21\lambda_3 - 108\lambda_4 + 144\lambda_5 + 216\lambda_6 = 156$

(3) $16\lambda_2 - 3\lambda_3 = -12$

(4) $28388\lambda_1 + 588\lambda_2 - 10585\lambda_3 - 28188\lambda_4 + 6351\lambda_5 + 9198\lambda_6 = -4608$

(5) $271432\lambda_1 - 54168\lambda_2 + 19360\lambda_3 - 155232\lambda_4 + 11064\lambda_5 + 15372\lambda_6 = 260313$

(6) $1180\lambda_1 + 52\lambda_2 - 271\lambda_3 - 900\lambda_4 + 777\lambda_5 + 1170\lambda_6 = 384.$

This is a system of 6 linearly independent linear equations. The solution of these equations solves the problem of determining I_{11} in terms of the fundamental invariants.

§ 10. *Construction of Sets of Products of Differences of Sixteen Numbers corresponding to the Heptads attached to Eight Numbers.*

Sixteen quantities $1, 2, \cdot \cdot \cdot, 8, a, b, \cdot \cdot \cdot, h$ form products of differences of the type $12 \cdot 34 \cdot 56 \cdot 78 \cdot ab \cdot cd \cdot ef \cdot gh$ in

$$15 \times 13 \times 11 \times 9 \times 7 \times 5 \times 3 = 2027025 \text{ ways.}$$

Out of 16 numbers we can form 120 pairs 12. Each product contains 8 pairs. Consequently there are sets of 15 products involving all 120 pairs just once.

There is more than one type of such sets but we are interested only in sets of the following symmetrical type ρ.

$$
\begin{array}{rll}
\rho = & 12\cdot34\cdot56\cdot78\cdot ab\cdot cd\cdot ef\cdot gh & p_1 \\
& 13\cdot24\cdot75\cdot86\cdot ac\cdot bd\cdot ge\cdot hf & p_2 \\
& 14\cdot32\cdot85\cdot67\cdot ad\cdot cb\cdot he\cdot fg & p_3 \\
& 15\cdot62\cdot37\cdot84\cdot ae\cdot fb\cdot cg\cdot hd & p_4 \\
& 16\cdot25\cdot74\cdot83\cdot af\cdot bc\cdot hc\cdot gd & \\
& 17\cdot82\cdot46\cdot53\cdot ag\cdot hb\cdot ec\cdot df & \\
& 18\cdot27\cdot36\cdot45\cdot ah\cdot bg\cdot cf\cdot de & \\
& 1a\cdot2b\cdot3c\cdot4d\cdot5e\cdot6f\cdot7g\cdot8h & \\
& 1b\cdot2a\cdot3d\cdot4c\cdot5f\cdot6e\cdot7h\cdot8g & \\
& 1c\cdot2d\cdot3a\cdot4b\cdot5g\cdot6h\cdot7e\cdot8f & \\
& 1e\cdot2f\cdot3g\cdot4h\cdot5a\cdot6b\cdot7c\cdot8d & \\
& 1d\cdot2c\cdot3b\cdot4a\cdot5h\cdot6g\cdot7f\cdot8e & \\
& 1f\cdot2e\cdot3h\cdot4g\cdot5b\cdot6a\cdot7d\cdot8c & \\
& 1g\cdot2h\cdot3e\cdot4f\cdot5c\cdot6d\cdot7a\cdot8b & \\
& 1h\cdot2g\cdot3f\cdot4e\cdot5d\cdot6c\cdot7b\cdot8a. &
\end{array}
$$

We may note that ρ is built up out of heptads in the same way that the heptads are built up out of triples of the type $12\cdot34$, $13\cdot24$, $14\cdot23$.

These sets are differentiated from other sets of 15 by the fact that they possess the property that if any two products contain 4 pairs of factors belonging to a heptad the set will contain the heptad determined by these. For example in the typical set ρ the products p_1 and p_2 contain the partial products

$$12\cdot34\cdot56\cdot78, \quad 13\cdot24\cdot75\cdot86.$$

Hence it contains the complete heptad determined by these two products.

In the case of eight points we saw that two products were sufficient to determine a heptad. In the case of 16 points we shall see that 4 products are sufficient to determine the set ρ. The products p_1 and p_2 contain the partial products

$$\left.\begin{array}{l} 12\cdot34\cdot56\cdot78 \\ 13\cdot24\cdot75\cdot86 \end{array}\right\} (1) \qquad \left.\begin{array}{l} 12\cdot34\cdot ab\cdot cd \\ 13\cdot24\cdot ac\cdot bd \end{array}\right\} (2)$$

So our set ρ must contain the two heptads determined by (1) and (2). The

product p_3 is determined uniquely by p_1 and p_2. If now we fill the fourth place in our set with p_4, ρ is completely determined, for p_1 and p_4 require that ρ contain the heptad determined by

$$\left.\begin{array}{l} 12\cdot 56\cdot ab\cdot ef \\ 15\cdot 62\cdot ae\cdot fb \end{array}\right\} (3)$$

while p_3 and p_4 require that ρ contain the heptad

$$\left.\begin{array}{l} 13\cdot 57\cdot ac\cdot eg \\ 17\cdot 53\cdot ag\cdot ec \end{array}\right\} (4)$$

The 4 heptads (1) $\cdots$ (4) completely determine ρ, for if we now attempt to fill the fifth place in the set we see that we must have

$$p_5 = 16\cdot 25\cdot 83\cdot 74\cdot af\cdot be\cdot hc\cdot gd.$$

Thus the complete set ρ is determined by the four products p_1, p_2, p_3, p_4. In addition to the heptads (1), $\cdots$, (4) the set ρ contains 11 others which occur as a consequence of the occurrence of (1), $\cdots$, (4). So every set ρ contains 15 complete heptads. There are really 30 heptads but we are restricting ourselves to heptads on the point 1, since every such heptad pairs off with one on the remaining eight numbers.

Let us now determine the number of the sets ρ. To do this we first determine the number of sets which contain any one heptad say

$$\sigma_1 = 12\cdot 34\cdot 56\cdot 78 \ (\equiv \pi_1),\ 13\cdot 24\cdot 57\cdot 68 \ (\equiv \pi_2),\ \cdots,\ \pi_7.$$

Now π_1 may be multiplied by any one of 105 products of the type

$$ab\cdot cd\cdot ef\cdot gh \equiv \pi_1'.$$

Now π_1' is associated in two heptads with 12 products. Consequently we may multiply π_2 by any one of 12 products in order to get p_2. Let us select $\pi_2' = ac\cdot bd\cdot ge\cdot hf$. π_1' and π_2' determine the heptad on the division $ab\cdot cd\cdot ef\cdot gh$ which is to occur. As we have seen p_3 is determined by p_1 and p_2. So we are left with 4 possible products π_4' by which π_4 may be multiplied in order to get p_4. We then have no further choice since p_1, p_2, p_3, p_4 completely determine ρ. Hence the number of sets containing the heptad σ_1 is

$$105 \times 12 \times 4 = 5040 = 7\,!$$

Now on point (1) there are 6435 sets of eight. On each such set there are

30 heptads. Hence $6435 \times 30 =$ total number of heptads formed from 16 quantities. Each heptad occurs 5040 times; but in each set there are 15 heptads. Hence if N denotes the number of sets ρ we have

$$N = \frac{6435 \times 30 \times 5040}{15} = 64864800.$$

We shall denote this system of N sets ρ by the symbol R.

In the heptad system we saw that every product $12 \cdot 34 \cdot 56 \cdot 78$ occurs twice. In the system R every product $12 \cdot 34 \cdot 56 \cdot 78 \cdot ab \cdot cd \cdot ef \cdot gh$ occurs 480 times since R is composed of 64864800 sets ρ and the number of distinct products is only 2027025. Consider now the summation

$$\sum^{2027025} (12 \cdot 34 \cdot 56 \cdot 78 \cdot ab \cdot cd \cdot ef \cdot gh)^{2i+1} = \sum p_i^{2i+1},$$

where every term of the summation is arranged positively according to the standard positive term

$$12 \cdot 34 \cdot 56 \cdot 78 \cdot ab \cdot cd \cdot ef \cdot gh.$$

Applying the same argument that we applied to the $\sum^{105} (12 \cdot 34 \cdot 56 \cdot 78)^{2i+1}$ we find that

$$\sum p_i^{2i+1} = 0 \quad \text{for } i = 0, 1, 2, 3, 4, 5, 6.$$

But $\sum p_i^{15} = \mu' \sqrt{\Delta}$ where Δ is the discriminant of the sixteenic whose roots are $1, 2, \cdots, a, \cdots, h$. As in the case of the heptads σ, if we add the identity and regard the products of ρ as substitutions we see that ρ is an Abelian group of order 16 under which ρ, considered as the set of products, is invariant.

§ 11. *On the Division of the System R into Two Sets.*

We have seen that any set ρ may be obtained from a heptad on eight of the sixteen quantities $1, \cdots, 8, a, \cdots, h$ by multiplying its products by the products of a second heptad on the remaining eight quantities. Any heptad, say σ_1, occurs in 5040 sets ρ in the system R. Of these, 2520 sets R_1 contain σ_1 with its products multiplied into the products of the set σ_a in a different order; and 2520 sets R_2, contain σ_1 with its product multiplied in every possible way into the product of the set s_a; where σ_a and s_a denote the 15σ- and $15s$-heptads built up out of the eight quantities a, b, c, d, e, f, g, h. Now there are 168 sets ρ in R_1 which contain σ_1 multiplied

in different orders into the heptad σ_{a_1} where σ_{a_1} is one of the 15 heptads σ_a. These 168 sets are sent into each other by even substitutions and are sent into the negative of the corresponding sets of 168 on s_{a_1} by odd substitutions on the symbols a, b, c, d, e, f, g, h. Any substitution which sends σ_a into $-s_a$ will send R_1 into R_2, since these substitutions involve only the quantities $a, \cdots, h$ and consequently leave σ_1 unaltered. The substitutions which send σ_a into $-s_a$ we know are of odd order. Hence we have the result that R_1 and R_2 are sent into themselves by even substitutions on $a, b, c, \cdots, h$ and R_1 is sent into the negative of R_2 by the odd substitutions.

Within R we have 6435 divisions R_1R_2 corresponding to the number of divisions of 16 things into sets of 8. Call a second such division R_1', R_2'. If one of the sets ρ in R_1 is sent into one of the sets ρ_1' of R_1' by even substitutions every set ρ in R_1 can be sent into every set ρ_1' of R_1' by even substitutions since the product of two even substitutions is even. On the other hand, R_2 is then sent into $-R_1'$ by odd substitutions and into R_2' by even substitutions on the 16 symbols, $1, 2, \cdots, a, \cdots, h$.

The argument requires expansion to constitute a proof, but it seems fairly certain that the system R divides into two sets which are each invariant under the even substitutions of $G_{16!}$ and are sent into the negative of the other by the odd substitutions.

The Borel Summability of Fourier Series *

By M. H. Stone.

In an attempt to apply a generalized Borel summation to the expansions arising from the differential system † $u''' + \lambda u = 0$, $u(0) = u(1) = u'(0) = 0$ it was found necessary to consider the corresponding question for Fourier series. From the investigations of C. N. Moore it was known that the ordinary Borel summation of Fourier series is relatively ineffective.‡ Consequently we modified the Borel sum in a simple manner to obtain the results which we shall describe in the present paper. The hope which prompted the study of this problem proved to be unfounded: the methods detailed below yielded no immediate advance in the consideration of the irregular expansion problem for which they were designed.

I. A Direct Method.

We shall denote by $K(x-y;\alpha)$ the sum of the series

$$2e^{\alpha\cos 2\pi(x-y)}\cos\alpha\sin 2\pi(x-y) - 1$$
$$= 1 + \sum_{m=1}^{m=\infty} \frac{2\alpha^m}{\Gamma(m+1)} \cos 2m\pi(x-y).$$

The Borel sum of the Fourier series on (0, 1) for an arbitrary summable function § $f(x)$ is defined as the limit for $A \to \infty$, when it exists, of the integral

$$\int_0^A e^{-\alpha} \int_0^1 f(y)\, K(x-y;\alpha)\, dy\, d\alpha.$$

We replace this integral by another,

$$\int_0^A (1-\alpha/A)e^{-\alpha} \int_0^1 f(y)\, K(x-y;\alpha)\, dy\, d\alpha,$$

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* Presented to the American Mathematical Society, October 31, 1925.

† Hopkins, *Transactions of the American Mathematical Society*, Vol. 20 (1919), pp. 245-259.

‡ C. N. Moore, *Proceedings of the National Academy*, Vol. 11, No. 6, pp. 284-287.

§ There should arise no confusion from the use of this expression to indicate a function integrable in the sense of Lebesgue.

in which we allow A to become infinite. In essence we superimpose a Cesàro sum of order one upon the Borel sum *; we should expect to be able to deal with Fourier series as effectively by this composite sum as by the Cesàro sum of order one. That such is the case is the main result of this paper.

We shall suppose that the function $f(x)$ is defined for all values of x as periodic of period one. A few simple and familiar reductions then show us that

$$\int_0^1 f(y)\,K(x-y;\,a)\,dy = \int_{-\frac{1}{2}+x}^{x} f(y)\,K(x-y;a)\,dy$$

$$+\int_x^{\frac{1}{2}+x} f(y)\,K(x-y;a)\,dy = \int_0^{\frac{1}{2}} [f(x+t)+f(x-t)]\,K(t;a)\,dt,$$

$$\int_0^1 K(x-y;a)\,dy = 1\,.$$

It follows that we can limit our consideration to that of the integral

$$\int_0^{\frac{1}{2}} \phi(t) \int_0^A (1-a/A)e^{-a}K(t;\,a)\,da\,dt \quad ,$$
$$\phi(t) = f(x+t)+f(x-t)-2f(x).$$

The integral with respect to a satisfies the following relations:

$$\left|\int_0^A (1-a/A)e^{-a}K(t;\,a)\,da\right| \leq MA \quad , \qquad 0 \leq t \leq \tfrac{1}{2} \quad ,$$

$$\int_0^A (1-a/A)e^{-a}K(t;a)\,da = 1/A\,\Bigg[\frac{\sin^2 2\pi t}{2(1-\cos 2\pi t)^2} +$$

$$\frac{\sin 2\pi t}{1-\cos 2\pi t}\;\frac{e^{A(\cos 2\pi t-1)}\{(\cos 2\pi t-1)\sin A\sin 2\pi t - \sin 2\pi t.\cos A\sin 2\pi t\}}{2(1-\cos 2\pi t)}$$

$$+\;\frac{e^{A(\cos 2\pi t-1)}\{(\cos 2\pi t-1)\cos A\sin 2\pi t + \sin 2\pi t.\sin A\sin 2\pi t\}}{1-\cos 2\pi t}$$

$$+\,2-e^{-a}\Bigg] = \frac{m(t,A)}{At^2} \quad , \qquad |\,m(t,A)\,| \leq M\,,\ 0 \leq t \leq \tfrac{1}{2}\ , \quad 0 \leq A.$$

As a result of these inequalities we can apply the reasoning of a paper of Hardy, slightly modified in a manner which we have pointed out elsewhere, to obtain the desired theorem.†

* Hardy and Riesz, "The General Theory of Dirichlet's Series," Cambridge Tracts, No. 18, 1915, p. 23.

† Hardy, *Proceedings of the London Mathematical Society*, Ser. 2, Vol. 12 (1913), pp. 365-372. The minor modifications of Hardy's procedure appear under Theorem XIV of a paper entitled "A Comparison of the Series of Fourier and Birkhoff" recently accepted by the *Transactions of the American Mathematical Society.*

THEOREM I. *If $f(x)$ is an arbitrary summable function of period one, then*

$$\lim_{A\to\infty} \int_0^A (1-a/A)\, e^{-a} \int_0^1 f(y)\, K(x-y\,;a)\,dy\,da = f(x)$$

almost everywhere, $0 \leq x \leq 1$; and if the function is continuous, the convergence is uniform, $0 \leq x \leq 1$.

II. THE DISCUSSION FOR A DIFFERENTIAL SYSTEM.

It is well-known that the Fourier series on the interval (0, 1) can be defined in terms of the differential system

$$u' + \lambda u = 0, \qquad u(0) - u(1) = 0.$$

By the use of $G(x, y; \lambda)$, the Green's function for this differential system, we shall set up a function $\Phi(x, y; a)$ which generalizes the function $K(x-y; a)$ of the preceding section. We let C_1 and C_2 be two contours in the λ-plane. C_1 is an analytic quadrilateral bounded by segments of two rays emanating from the origin at an angle β with the positive axis of imaginaries and by arcs of two circles with centers at the origin and radii λ_0 and Λ respectively. The circular portions of C_1 on the first quadrant we call γ_1 and γ_3, the rectilinear portion γ_2, the enumeration proceeding in a counter-clockwise direction around C_1. The part of C_1 on the second quadrant is similarly composed of γ_4, γ_5, and γ_6. C_2 is symmetric with C_1 in the real axis. Its component arcs we denote by γ_7, γ_8, γ_9, γ_{10}, γ_{11}, and γ_{12}, beginning in the third quadrant, and numbering in a counter-clockwise direction. The magnitude of the angle $\beta > 0$ shall be determined subsequently, the radius λ_0 shall be restricted so that $0 < \lambda_0 < 2\pi$, and the radius Λ shall satisfy the inequality $|\Lambda - 2n\pi| \geqq \epsilon > 0$ for all integral n. Since the function $G(x,y;\lambda)$ has a simple pole with residue $e^{2n\pi i(y-x)}$ at the point $\lambda = 2n\pi i$ for n a positive or negative integer or zero, and is analytic elsewhere, the meaning of the function

$$\Phi(x, y; a; \Lambda) = 1 + 1/2\pi i \int_{C_1} \frac{a^{-\lambda i}}{\Gamma(-\lambda i + 1)}\, G\, d\lambda \; + \; 1/2\pi i \int_{C_2} \frac{a^{\lambda i}}{\Gamma(\lambda i + 1)}\, G d\lambda$$

is at once clear. On allowing Λ to become infinite, we obtain

$$\Phi(x, y; a) = \lim_{\Lambda\to\infty} \Phi(x, y; a; \Lambda).$$

The method of summing Fourier series which we shall discuss in the present section is based upon a study of the integral

$$\int_0^A (1-a/A)\, e^{-a} \int_0^1 f(y)\, \Phi(x, y; a)\, dy\, da.$$

When $A \rightarrow \infty$ in this integral, we have a generalization of the summatory process of the preceding section.*

Before passing to the convergence proof, we shall introduce the useful notation $E(x, y, a)$ for a function of x, y, a continuous $0 < a \leqq x \leqq b < 1$, $0 \leqq y \leqq 1$, $0 \leqq a < \infty$, such that $\int_0^\infty e^{-a} E(x, y, a)\, da$ is a uniformly convergent integral for the ranges of x and y in question. For any function $E(x, y, a)$ we have also the property †

$$\lim_{A\rightarrow\infty} \int_0^A (1-a/A) e^{-a} E(x, y, a)\, da = \int_0^\infty e^{-a} E(x, y, a)\, da.$$

The proof of the second theorem is based upon two lemmas, which we shall now prove.

LEMMA I. *If $\Phi(x, y; a)$ is the function defined above, then*

$$\Phi(x, y; a) = 1/\pi \int_{\lambda_0}^\infty \frac{a^\lambda}{\Gamma(\lambda+1)} \cos \lambda(x-y)\, d\lambda + E(x, y, a).$$

On substituting in the formula for $\Phi(x, y; a; \Lambda)$ from the equations

$$G(x, y; \lambda) = \{e^{\lambda(y-x)};\ 0\} + \frac{e^{\lambda(y-1)-\lambda x}}{1-e^{-\lambda}}\ ,\ \mathcal{R}(\lambda) \geqq 0\,,$$

$$G(x, y; \lambda) = \{0; -e^{\lambda(y-x)}\} - \frac{e^{\lambda y + \lambda(1-x)}}{1-e^{\lambda}}\ ,\ \mathcal{R}(\lambda) \leqq 0\,,$$

where $\mathcal{R}(\ )$ means "the real part of," and $\{A; B\}$ denotes the function A for $x \geqq y$, the function B for $x \leqq y$, we find

$$\Phi(x, y; a; \Lambda) = 1 + 1/2\pi i \int_{\lambda_0 i}^{\Lambda i} \frac{a^{-\lambda i}}{\Gamma(-\lambda i + 1)} e^{\lambda(y-x)}\, d\lambda - 1/2\pi i \int_{-\lambda_0 i}^{-\Lambda i} \frac{a^{\lambda i}}{\Gamma(\lambda i + 1)} e^{\lambda(y-x)}\, d\lambda$$

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* Marcel Riesz, *Acta Mathematica*, Vol. 35 (1912), pp. 253-270.

† Hardy and Riesz, *loc. cit.*

$$+\,1/2\pi i \int_{\gamma_1+\gamma_2+\gamma_3} \frac{a^{-\lambda i}}{\Gamma(-\lambda i+1)}\,\frac{e^{-\lambda(1-y+x)}}{1-e^{-\lambda}}\,d\lambda\,-$$

$$1/2\pi i \int_{\gamma_4+\gamma_5+\gamma_6} \frac{a^{-\lambda i}}{\Gamma(-\lambda i+1)}\,\frac{e^{\lambda(y+1-x)}}{1-e^{\lambda}}\,d\lambda$$

$$-\,1/2\pi i \int_{\gamma_7+\gamma_8+\gamma_9} \frac{a^{\lambda i}}{\Gamma(\lambda i+1)}\,\frac{e^{\lambda(y+1-x)}}{1-e^{-\lambda}}\,d\lambda\,+$$

$$1/2\pi i \int_{\gamma_{10}+\gamma_{11}+\gamma_{12}} \frac{a^{\lambda i}}{\Gamma(\lambda i+1)}\,\frac{e^{-\lambda(y+1-x)}}{1-e^{\lambda}}\,d\lambda$$

by an application of Cauchy's theorem to the integrals involving bracketed terms. An obvious change of variables in the first two integrals yields

$$1/2\pi i \int_{\lambda_0 i}^{\Lambda i} \frac{a^{-\lambda i}}{\Gamma(-\lambda i+1)}\,e^{\lambda(y-x)}d\lambda - 1/2\pi i \int_{-\lambda_0 i}^{-\Lambda i} \frac{a^{\lambda i}}{\Gamma(\lambda i+1)}\,e^{\lambda(y-x)}d\lambda =$$

$$1/2\pi \int_{\lambda_0}^{\Lambda} \frac{a^{\lambda}}{\Gamma(\lambda+1)}\,e^{-\lambda i(x-y)}d\lambda + 1/2\pi \int_{\lambda_0}^{\Lambda} \frac{a^{\lambda}}{\Gamma(\lambda+1)}\,e^{\lambda i(x-y)}d\lambda =$$

$$1/\pi \int_{\lambda_0}^{\Lambda} \frac{a^{\lambda}}{\Gamma(\lambda+1)}\,\cos\lambda(x-y)\,d\lambda.$$

We shall next discuss as typical of the remaining integrals that one taken over the arcs γ_1, γ_2, γ_3. Since γ_1 is a fixed finite arc we have at once

$$\int_0^A e^{-a} \int_{\gamma_1} \frac{a^{-\lambda i}}{\Gamma(-\lambda i+1)}\,\frac{e^{-\lambda(1-y+x)}}{1-e^{-\lambda}}\,d\lambda da =$$

$$\int_{\gamma_1} \frac{e^{-\lambda(1-y+x)}}{1-e^{-\lambda}} \int_0^A \frac{e^{-a}a^{-\lambda i}}{\Gamma(-\lambda i+1)}\,dad\lambda \;\to\; \int_{\gamma_1} \frac{e^{-\lambda(1-y+x)}}{1-e^{-\lambda}}\,d\lambda$$

as $A \to \infty$. In short,

$$1/2\pi i \int_{\gamma_1} \frac{a^{-\lambda i}}{\Gamma(-\lambda i+1)}\,\frac{e^{-\lambda(1-y+x)}}{1-e^{-\lambda}}\,d\lambda = E(x,y,a).$$

Next, on γ_3 we have $\lambda = \Lambda i e^{-i\theta}$, $0 \leq \theta \leq \beta < \pi/2$. For this range of values

$$\left|\frac{e^{-\lambda(1-y+x)}}{1-e^{-\lambda}}\right| \leq K_1\,, \qquad |d\lambda| = \Lambda\,d\theta\,,$$

$$|a^{-\lambda i}| = |e^{\Lambda e^{-i\theta}\log a}| = e^{\Lambda\cos\theta\,.\,\log a}\,,$$

$$|\Gamma(-\lambda i+1| = |\Gamma(\Lambda e^{-i\theta}+1)| \geq 1/K_2\ |e^{(\Lambda e^{-i\theta}+\frac{1}{2})\log\Lambda e^{-i\theta}-\Lambda e^{-i\theta}}| =$$

$$= 1/K_2\ e^{\Lambda\cos\theta\,.\,\log\Lambda+\frac{1}{2}\log\Lambda-\theta\Lambda\sin\theta-\Lambda\cos\theta}$$

$$\geq 1/K_2\ e^{\Lambda\cos\theta\,.\,\log\Lambda-\theta\Lambda\sin\theta-\Lambda\cos\theta}\,.$$

Hence

$$\left| \int_{\gamma_3} \frac{a^{-\lambda i}}{\Gamma(-\lambda i+1)} \; \frac{e^{-\lambda(1-y+x)}}{1-e^{-\lambda}} \, d\lambda \right| \leq K_1 K_2 \int_0^\beta e^{\Lambda \cos\theta \,.\, \log \frac{ae}{\Lambda} + \Lambda\theta \sin\theta} \, \Lambda d\theta$$

$$\leq K_1 K_2 \, \beta \Lambda e^{\Lambda \cos\beta \,.\, \log \frac{ae}{\Lambda} + \Lambda\beta \sin\beta} \to 0$$

as $\Lambda \to \infty$. Then on γ_2 we have $\lambda = \lambda' i e^{-i\beta}$, $\lambda_0 \leq \lambda' \leq \Lambda$. For this range of variables we see that

$$\left| \frac{e^{-\lambda(1-y+x)}}{1-e^{-\lambda}} \right| \leq \frac{e^{-\lambda' \sin\beta \,.\, (1-y+x)}}{|1-e^{-\lambda}|} \leq K_1 e^{-\lambda' \sin\beta \,.\, (1-y+x)} ,$$

$$|a^{-\lambda i}| = e^{\lambda' \cos\beta \,.\, \log a} , \quad |d\lambda| = d\lambda',$$

$$|\Gamma(-\lambda i + 1)| \leq 1/K_2 \, e^{\lambda' \cos\beta \,.\, \log \lambda' - \beta\lambda' \sin\beta - \lambda' \cos\beta} .$$

Hence

$$\left| \int_{\gamma_2} \frac{a^{-\lambda i}}{\Gamma(-\lambda i+1)} \; \frac{e^{-\lambda(1-y+x)}}{1-e^{-\lambda}} \, d\lambda \right| \leq K_1 K_2 \int_{\lambda_0}^{\Lambda} e^{\lambda' \cos\beta \,.\, \log \frac{ae}{\lambda'} + \lambda'[\beta-(1-y+x)] \sin\beta} \, d\lambda'$$

In this integral the function $e^{\lambda' \cos\beta \,.\, \log \frac{ae}{\lambda'}}$ takes on its maximum value $e^{a \cos\beta}$ for $\lambda' = a$, the positive real root of the equation

$$\frac{d}{d\lambda'} \, e^{\lambda' \cos\beta \,.\, \log \frac{ae}{\lambda'}} = e^{\lambda' \cos\beta \,.\, \log \frac{ae}{\lambda'}} (\cos\beta \cdot \log ae/\lambda' - \cos\beta) = 0.$$

We can determine $\beta > 0$ so that $[\beta - (1 - y + x)] \sin\beta \leq -\epsilon < 0$ for $0 < a \leq x \leq b < 1$ and $0 \leq y \leq 1$. Thus it is clear that

$$0 \leq \int_{\lambda_0}^{\Lambda} e^{\lambda' \cos\beta \,.\, \log \frac{ae}{\lambda'} + \lambda'[\beta-(1-y+x)] \sin\beta} \, d\lambda' \leq e^{a \cos\beta} \int_{\lambda_0}^{\infty} e^{-\lambda' \epsilon} \, d\lambda' .$$

The immediate consequence of this result is that as $\Lambda \to \infty$

$$1/2\pi i \int_{\gamma_3} \frac{a^{-\lambda i}}{\Gamma(-\lambda i+1)} \; \frac{e^{-\lambda(1-y+x)}}{1-e^{-\lambda}} \, d\lambda \to E\,(x, y, a) \quad .$$

Thus we find

$$\Phi(x, y ; a) = \lim_{\Lambda \to \infty} \; \Phi(x, y ; a ; \Lambda)$$

$$1/\pi \int_{\lambda_0}^{\infty} \frac{a^{\lambda}}{\Gamma(\lambda+1)} \; \cos\lambda(x-y) \, d\lambda + E(x, y, a)$$

as we were to show.

LEMMA II. *The function of x, y, a defined by the infinite integral*

$$1/\pi \int_{\lambda_0}^{\infty} \frac{a^\lambda}{\Gamma(\lambda+1)} \cos \lambda(x-y)\, d\lambda$$

is expressible in the form

$$1/\pi\, e^{a \cos (x-y)} \cos a \sin (x-y) - 1/2\pi + E(x, y, a).$$

We first obtain a representation of the integrand in a more convenient form. In the theory of Bessel functions it is shown that

$$\frac{a^\lambda}{\Gamma(\lambda+1)} = \sum_{m=0}^{m=\infty} a^m/m!\ J_{\lambda+m}(2a)$$

uniformly $0 \leq a \leq A$, $0 \leq \lambda \leq \Lambda$; and that

$$J_{\lambda+m}(2a) = 1/\pi \int_0^{\pi/2} \cos([\lambda+m]\theta - 2a \sin\theta)\, d\theta$$
$$+ 1/\pi \int_0^{\infty} e^{-[\lambda+m]\theta} \sin(2a \cosh\theta - \pi/2[\lambda+m])\, d\theta$$

uniformly $0 \leq a \leq A$, $0 < \lambda_1 \leq \lambda \leq \lambda_2$.* Now it is apparent that the two series

$$e^{a\cos\theta} \cos(\lambda\theta - a \sin\theta) = \sum_{m=0}^{m=\infty} a^m/m!\ \cos([\lambda+m]\theta - 2a\sin\theta),$$

$$e^{-\lambda\theta} \sin(ae^\theta - \pi\lambda/2) = \sum_{m=0}^{m=\infty} a^m/m!\ e^{-[\lambda+m]\theta} \cdot \sin(2a\cosh\theta - \pi/2[\lambda+m])$$

converge uniformly in the variable θ on the ranges $0 \leq \theta \leq \pi/2$ and $0 \leq \theta < \infty$ respectively, for $a \geq 0$, $\lambda > 0$. Consequently

$$\frac{a^\lambda}{\Gamma(\lambda+1)} = 1/\pi \int_0^{\pi/2} e^{a\cos\theta} \cos(\lambda\theta - a\sin\theta)\, d\theta$$
$$+ 1/\pi \int_0^{\infty} e^{-\lambda\theta} \sin(ae^\theta - \pi\lambda/2)\, d\theta = 1/2\pi \int_{-\pi/2}^{+\pi/2} e^{a\cos\theta} \cos(\lambda\theta - a\sin\theta)\, d\theta$$
$$+ 1/\pi \int_0^{\infty} e^{-\lambda\theta} \sin(ae^\theta - \pi\lambda/2)\, d\theta\ .$$

With this identity at our disposal we are prepared to discuss the lemma. First we observe that

$$1/\pi \int_0^{\lambda_0} \frac{a^\lambda}{\Gamma(\lambda+1)} \cos\lambda(x-y)\, d\lambda = E(x, y, a)$$

* Watson, *Theory of Bessel Functions*, Cambridge, 1922, § 5. 22, § 6. 20.

since we have

$$\lim_{A\to\infty} \int_0^A e^{-a} \int_0^{\lambda_0} \frac{a^\lambda}{\Gamma(\lambda+1)} \cos\lambda(x-y)\,d\lambda da =$$

$$\lim_{A\to\infty} \int_0^{\lambda_0} \cos\lambda(x-y) \int_0^A \frac{e^{-a}a^\lambda}{\Gamma(\lambda+1)}\,da d\lambda = \int_0^{\lambda_0} \cos\lambda(x-y)\,d\lambda.$$

Thus the truth of the lemma depends upon the behavior of the integral

$$1/\pi \int_0^\infty \frac{a^\lambda}{\Gamma(\lambda+1)} \cos\lambda t d\lambda$$

where we have replaced $x-y$ by t. For the second integral in the identity for $\dfrac{a^\lambda}{\Gamma(\lambda+1)}$ we see that

$$\int_{\lambda_1}^{\lambda_2}\int_0^\infty e^{-\lambda\theta}\sin(ae^\theta-\pi\lambda/2)\cos\lambda t\,d\theta\,d\lambda =$$

$$\int_0^\infty\int_{\lambda_1}^{\lambda_2} e^{-\lambda\theta}\sin(ae^\theta-\pi\lambda/2)\cos\lambda t\,d\lambda\,d\theta =$$

$$= \int_0^\infty \frac{\sin ae^\theta}{2}\Bigg[e^{-\lambda\theta}\Bigg\{\frac{-\theta\cos\lambda[t-\pi/2]+[t-\pi/2]\sin\lambda[t-\pi/2]}{\theta^2+[t-\pi/2]^2} +$$

$$+ \frac{-\theta\cos\lambda[t+\pi/2]+[t+\pi/2]\sin\lambda[t+\pi/2]}{\theta^2+[t+\pi/2]^2}\Bigg\}\Bigg]_{\lambda=\lambda_1}^{\lambda=\lambda_2} d\theta +$$

$$+ \int_0^\infty \frac{\cos ae^\theta}{2}\Bigg[e^{-\lambda\theta}\Bigg\{\frac{-\theta\sin\lambda[t-\pi/2]-[t-\pi/2]\cos\lambda[t-\pi/2]}{\theta^2+[t-\pi/2]^2} +$$

$$+ \frac{+\theta\sin\lambda[t+\pi/2]+[t+\pi/2]\cos\lambda[t+\pi/2]}{\theta^2+[t+\pi/2]^2}\Bigg\}\Bigg]_{\lambda=\lambda_1}^{\lambda=\lambda_2} d\theta$$

$$\to \tfrac{1}{2}\int_0^\infty \sin ae^\theta\left[\frac{\theta}{\theta^2+(t-\pi/2)^2}+\frac{\theta}{\theta^2+(t+\pi/2)^2}\right]d\theta +$$

$$+ \tfrac{1}{2}\int_0^\infty \cos ae^\theta\left[\frac{t-\pi/2}{\theta^2+(t-\pi/2)^2}-\frac{t+\pi/2}{\theta^2+(t+\pi/2)^2}\right]d\theta.$$

The limit for $\lambda_1\to 0$, $\lambda_2\to\infty$ can be obtained here by allowing $\lambda_1\to 0$, $\lambda_2\to\infty$, under the integral sign, because of the fact that the integrals

$$\int_0^\infty \begin{pmatrix}\sin ae^\theta\\ \cos ae^\theta\end{pmatrix}\cdot e^{-\lambda\theta}\cdot\begin{pmatrix}\dfrac{\theta}{\theta^2+k^2}\\ \dfrac{1}{\theta^2+k^2}\end{pmatrix}d\theta = \int_1^\infty \begin{pmatrix}\sin a\mu\\ \cos a\mu\end{pmatrix}\cdot\mu^{-1-\lambda}\cdot\begin{pmatrix}\dfrac{\log\mu}{(\log\mu)^2+k^2}\\ \dfrac{1}{(\log\mu)^2+k^2}\end{pmatrix}d\mu$$

are uniformly convergent $0\leq\lambda<\infty$, for fixed $a>0$. To show that this limiting value is of the form $E(x,y,a)$ we note that

$$\left|\int_0^\infty \cos ae^\theta\frac{d\theta}{\theta^2+k^2}\right| \leq \int_0^\infty\frac{d\theta}{\theta^2+k^2},\quad 0\leq a<\infty,$$

and that an integration by parts shows that

$$\left| \int_0^\infty \sin ae^\theta \frac{\theta\, d\theta}{\theta^2+k^2} \right| = \left| 1/a \int_0^\infty \cos ae^\theta \frac{d}{d\theta} \frac{\theta e^{-\theta}}{\theta^2+k^2}\, d\theta \right|$$

$$\leq 1/a \int_0^\infty \left| \frac{d}{d\theta} \frac{\theta e^{-\theta}}{\theta^2+k^2} \right| d\theta \leq k/a\,,$$

$$0 < a < \infty\,.$$

Thus we have

$$\int_0^\infty \int_0^\infty e^{-\lambda\theta} \sin(ae^\theta - \pi\lambda/2) \cos\lambda(x-y)\, d\theta d\lambda = E(x,y,a).$$

We now turn to the first integral. Since this integral is valid for $0 \leq \lambda \leq \lambda_1$ we find

$$\int_0^{\lambda_1} \int_{-\pi/2}^{+\pi/2} e^{a\cos\theta} \cos(\lambda\theta - a\sin\theta) \cos\lambda t\, d\theta\, d\lambda =$$

$$\int_{-\pi/2}^{+\pi/2} \int_0^{\lambda_1} e^{a\cos\theta} \cos(\lambda\theta - a\sin\theta) \cos\lambda t\, d\lambda\, d\theta =$$

$$= \tfrac{1}{2} \int_{-\pi/2}^{+\pi/2} e^{a\cos\theta} \Big[\frac{\sin(\lambda_1[\theta - t] - a\sin\theta) + \sin a\sin\theta}{\theta + t} +$$

$$+ \frac{\sin(\lambda_1[\theta - t] - a\sin\theta) + \sin a\sin\theta}{\theta - t} \Big] d\theta =$$

$$= \int_{-\pi/2}^{+\pi/2} e^{a\cos\theta} \frac{\sin(\lambda_1[\theta - t] - a\sin\theta) + \sin a\sin\theta}{\theta - t}\, d\theta =$$

$$= \int_{-\pi/2}^{+\pi/2} e^{a\cos\theta} \cos a\sin\theta \frac{\sin\lambda_1[\theta - t]}{[\theta - t]}\, d\theta$$

$$+ \int_{-\pi/2}^{+\pi/2} e^{a\cos\theta} \sin a\sin\theta \frac{1 - \cos\lambda_1[\theta - t]}{[\theta - t]}\, d\theta.$$

The first integral of this sum has the limit $\pi e^{a\cos t} \cos a\sin t$ since it is essentially the familiar Dirichlet integral. The second we can write as

$$\mathcal{R} \int_{-\pi/2}^{+\pi/2} e^{a\cos\theta} \sin a\sin\theta \frac{1 - e^{i\lambda_1(\theta - t)}}{\theta - t}\, d\theta\,.$$

We let C be the semi-circle $\theta = \dfrac{\pi e^{i\phi}}{2}$, $0 \leq \phi \leq \pi$, in the plane of the complex variable θ. By Cauchy's theorem

$$\int_{-\pi/2}^{+\pi/2} e^{a\cos\theta} \sin a\sin\theta \frac{1 - e^{i\lambda_1(\theta - t)}}{\theta - t}\, d\theta = \int_C e^{a\cos\theta} \sin a\sin\theta \frac{1 - e^{i\lambda_1(\theta - t)}}{\theta - t}\, d\theta.$$

On C we have for fixed a, and for all t, $-1+\epsilon \leq t \leq 1-\epsilon$, $\epsilon \geq 0$

$$\left| \frac{e^{a\cos\theta}\sin a\sin\theta}{\theta - t} \quad d\theta \right| \leqq K\,d\phi,$$

$$| e^{i\lambda_1(\theta-t)} | = | e^{i\lambda_1\theta} | = e^{i\lambda_1\pi \sin\phi/2},$$

so that

$$\left| \int_C e^{a\cos\theta}\sin a\sin\theta \,\frac{e^{i\lambda_1(\theta-t)}}{\theta-t} d\theta \right| \leq K \int_0^\pi e^{-\lambda_1\pi\sin\phi/2}\, d\phi \to 0$$

and

$$\mathcal{R}\int_{-\pi/2}^{+\pi/2} e^{a\cos\theta}\sin a\sin\theta \,\frac{1-e^{i\lambda_1(\theta-t)}}{\theta-t}\, d\theta \to \mathcal{R}\int_C \frac{e^{a\cos\theta}\sin a\sin\theta}{\theta-t}\, d\theta$$

as $\lambda_1 \to \infty$. It remains for us to prove that the last integral obtained is of the form $E(x, y, a)$. Now

$$\int_0^A e^{-a}\int_C e^{a\cos\theta}\sin a\sin\theta\,\frac{d\theta}{\theta-t}\, da =$$

$$= \int_C \int_0^A \frac{e^{a\cos(\theta-1)}\sin a\sin\theta}{\theta-t}\, da d\theta =$$

$$\int_C \frac{e^{A(\cos\theta-1)}\left[(\cos\theta-1)\sin A\sin\theta - \sin\theta\cdot\cos A\sin\theta\right]}{2(1-\cos\theta)}\, d\theta$$

$$+\int_C \frac{\sin\theta}{2(1-\cos\theta)}\,\frac{d\theta}{\theta-t}.$$

The first integral in this sum satisfies the inequalities

$$\left| \int_C e^{A(\cos\theta-1)}\,\frac{\left[(\cos\theta-1)\sin A\sin\theta - \sin\theta\cdot\cos A\sin\theta\right]}{2(1-\cos\theta)\,(\theta-t)}\, d\theta \right| \leq$$

$$K/\sqrt{2}\int_0^\pi \left[\, | e^{A(\cos\theta-1)}\sin A\sin\theta | + | e^{A(\cos\theta-1)}\cos A\sin\theta | \,\right]_C d\phi \leq$$

$$K\int_0^\pi | e^{A(\cos\theta-1+i\sin\theta)} |_C\, d\phi.$$

The last inequality here depends upon the fact that

$$|a| + |b| \leq \sqrt{2}\,|a+bi|.$$

For $\theta = \pi e^{i\phi}/2$, $0 \leq \phi \leq \pi$, we find

$$| e^{A(\cos\theta-1)+Ai\sin\theta} | = | e^{A(e^{i\theta}-1)} | \;\; | e^{A(e^{i\pi e^{i\phi}/2}-1)} |$$

$$= | e^{A(e^{i\pi/2\cos\phi - \pi/2\sin\phi}-1)} |$$

$$= | e^{A(e^{-\pi/2\sin\phi}[\cos\pi/2\cos\phi + i\sin\pi/2\cos\phi]-1)} |$$

$$= | e^{A(e^{-\pi/2\sin\phi}\cos\pi/2\cos\phi-1)} |.$$

Hence

$$\int_0^{\pi} \mid e^{A(\cos\theta-1)+Ai\sin\theta} \mid_C d\phi = \int_0^{\pi} e^{A(e^{-\pi/2\sin\phi}\cos\pi/2\cos\phi-1)} d\phi \to 0$$

as $A \to \infty$. Consequently

$$\lim_{A\to\infty} \int_0^A e^{-a} \mathcal{R} \int_C \frac{e^{a\cos\theta}\sin a\sin\theta}{\theta - t} d\theta da = \mathcal{R} \int_C \frac{\sin\theta}{2(1-\cos\theta)} \frac{d\theta}{\theta - t}$$

and

$$\mathcal{R} \int_C \frac{e^{a\cos\theta}\sin a\sin\theta}{\theta - t} d\theta = E(x, y, a) .$$

On combining the foregoing material, and noting that any constant is of the form $E(x, y, a)$, we see that

$$1/\pi \int_{\lambda_0}^{\infty} \frac{a^{\lambda}}{\Gamma(\lambda+1)} \cos\lambda(x-y)d\lambda = 1/\pi\, e^{a\cos(x-y)}\cos a\sin(x-y) - 1/2\pi + E(x, y, a)$$

as we were to show.

From these two lemmas and Theorem I, it is not difficult to prove the second theorem of the paper.

THEOREM II. *If $\Phi(x, y; a)$ is the function defined above, then*

$$\lim_{A\to\infty} \int_0^A (1-a/A)e^{-a} \int_0^1 f(y)\Phi(x, y, a)dy\, da = f(x)$$

almost everywhere, $0 < x < 1$. If the function $f(x)$ is continuous, $0 \leq x \leq 1$. then the convergence is uniform, $0 < a \leq x \leq b < 1$.

From Lemmas I and II we see that

$$\Phi(x, y; a) = 1/\pi\, e^{a\cos(x-y)}\cos a\sin(x-y) - 1/2\pi + E(x, y, a) .$$

Now since

$$\int_0^A (1-a/A)e^{-a} E(x, y, a)da$$

converges uniformly to a continuous function $Q(x, y)$, $0 < a \leq x \leq b < 1$, $0 \leq y \leq 1$, as $A \to \infty$, we can write

$$\lim_{A\to\infty} \int_0^A (1-a/A)e^{-a} \int_0^1 f(y)E(x, y, a)dyda = \int_0^1 f(y)Q(x, y)dy.$$

The contribution from that portion of $\Phi(x, y, a)$ given in explicit terms can be brought under Theorem I by setting

$$x = 2\pi x', \qquad y = 2\pi y',$$

$$F(y') = f(2\pi y'), \quad 0 \leq y' \leq 1/2\pi \ , \qquad F(y') = 0, \qquad 1/2\pi < y' < 1 ,$$

so that

$$\int_0^1 f(y)[1/\pi\, e^{a \cos (x-y)} \cos a \sin (x-y) dy = \int_0^1 F(y')K(x'-y';\, a)\, dy'.$$

Thus we see that

$$\lim_{A\to\infty} \int_0^A (1-a/A)e^{-a} \int_0^1 f(y)\Phi(x,y;a)\, dy da =$$

$$f(x) + \int_0^1 f(y)Q(x,y)\, dy$$

almost everywhere, $0 < a \leq x \leq b < 1$, the convergence being uniform if $f(x)$ is continuous, $0 \leq x \leq 1$.

By the definition of $\Phi(x,y;a)$ it is evident that

$$\int_0^1 \Phi(x,y;a)\, dy = 1, \int_0^1 \cos 2m\pi y\, \Phi(x,y;a)\, dy = \frac{a^{2m\pi}}{\Gamma(2m\pi+1)} \cos 2m\pi x,$$

$$\int_0^1 \sin 2m\pi y\, \Phi(x,y;a)\, dy = \frac{a^{2m\pi}}{\Gamma(2m\pi+1)} \sin 2m\pi x, \quad m = 1, 2, \cdots.$$

The substitution of these equations in the result just established gives us

$$\int_0^1 Q(x,y)\, dy = \lim_{A\to\infty} \int_0^A (1-a/A)e^{-a} da - 1 = 0 \quad ,$$

$$\int_0^1 \cos 2m\pi y\, Q(x,y)\, dy =$$

$$\lim_{A\to\infty} \int_0^A (1-a/A) \frac{e^{-a}a^{2m\pi}}{\Gamma(2m\pi+1)} \cos 2m\pi x da - \cos 2m\pi x$$

$$= 0,$$

$$\int_0^1 \sin 2m\pi y\, Q(x,y)\, dy = 0 \quad , \qquad 0 < a \leq x \leq b < 1.$$

By the closure of the set of functions 1, $\cos 2\pi x$, $\sin 2\pi x$, $\cdots$, we must have $Q(x, y) \equiv 0$, $0 < a \leq x \leq b < 1$, $0 \leq y \leq 1$, or, since a and b are arbitrary constants except for the inequality $0 < a < b < 1$, we have

$$Q(x, y) \equiv 0 \quad , \quad 0 < x < 1 \quad , \quad 0 \leq y \leq 1 \, .$$

Thus it is clear that

$$\lim_{A\to\infty} \int_0^A (1-a/A)e^{-a} \int_0^1 f(y)\, \Phi(x,y;a)\, dy da = f(x)$$

almost everywhere, $0 < x < 1$, the convergence being uniform, $0 < a \leq x \leq b < 1$, when $f(x)$ is continuous on the interval $(0, 1)$. This completes the theorem.

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Self Dual Space Curves.*

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1. *Definitions and statement of problems.* The only paper dealing with self dual space curves that I have been able to find is one by Steinmetz † in which he treats of special curves of this kind that exist in a linear nul-system in space.

The following symbols are to be used in this paper to denote the characteristics of a space curve of order n, class m, rank r and genus p: ‡

h number of apparent double points,
H number of nodes,
β number of cusps,
α number of stationary planes,
G number of double osculating planes,
g number of lines of intersection of two osculating planes that lie in a given plane,
v number of linear inflections,
ω number of bitangents,
x order of nodal curve of developable,
y class of nexal torse, that is, the number of planes, each containing two non-consecutive tangents to the curve, which pass through a given point. The envelope of all such planes is called the nexal torse by Cayley.
q class of nodal curve,
q' order of nexal torse,
k number of apparent double points of nodal curve,
k' number of apparent double plane of nexal torse,
γ number of cusps of nodal curve,
γ' number of stationary planes of nexal torse,

* Presented to the American Mathematical Society, February 27, 1926.

† Charles P. Steinmetz, "On the curves which are self-reciprocal in a linear nul-system and their configurations in space," *American Journal of Math.*, Vol. 14 (1892), pp. 161-186.

‡ This is Cayley's notation, with these exceptions: n and m are interchanged and p *is* used for the genus. For a table of the various notations, see Pascal's *Repertorio di Mat. Sup.*, Vol. II, p. 323.

t number of triple points of nodal curve,

t' number of triple tangent planes of nexal torse,

Of these, r, v, ω, are each self dual and the following pairs are duals: n, m; h, g; H, G; α, β; x, y; q, q'; k, k'; γ, γ'; t, t'.

If any seven of the above characteristics are given, the remaining fifteen are thereby determined. By a space curve type we shall mean a set of characteristics (either all or a sufficient number to determine all) considered as belonging to the same curve. Since a curve of given type always has a certain number of degrees of freedom, there are infinitely many curves of each curve type all of which have the same descriptive properties. Since only descriptive properties are to be considered in this paper, from this point on we shall use the term "curve" to mean "curve type."

A space curve is self dual if the value of each and every point characteristic of the curve and its dual plane characteristic is the same, that is if $n = m$, $h = g$, $H = G$, $\alpha = \beta$, $x = y$, $q = q'$, $k = k'$, $\gamma = \gamma'$, $t = t'$. This definition is equivalent to the one given by Steinmetz in the paper cited above.

The purpose of the present paper is to solve these four problems:

(I) To find whether, and if so under what conditions, each of the above nine equalities given in the definition is sufficient as well as necessary.

(II) To find limits for the singularities of self dual space curves.

(III) To find the conditions on a plane curve that it be the projection of a self dual space curve and the conditions on a space curve that its plane projection curve be self dual.

(IV) To find the conditions under which both the space curve and its plane projection curve can be self dual.

2. *Sufficient conditions for self dual space curves.* The conditions are stated in the following theorem whose proof follows.

THEOREM. For any space curve, of the following eight quantities, all of which are necessary for the curve to be self dual, the first four are also sufficient and the latter four are sufficient except for certain relations among the remaining characteristics that lead to definite curve types for certain values of n.

(1)	$n = m$,	(5)	$t = t'$,
(2)	$\alpha = \beta$,	(6)	$\gamma = \gamma'$,
(3)	$h + H = g + G$,	(7)	$q = q'$,
(4)	$x = y$,	(8)	$k = k'$.

The proofs of (1), (2) and (4) follow directly from the Cayley-Salmon formulas. In the case of (3), from these formulas there results the relation

$$(n-m)(n+m-7)=0.$$

Then when $m \neq n$, $m+n=7$. For all space cubics, however, $m=3$ so that no space curve exists for which $m+n=7$. Therefore condition (3) is also sufficient.

If a space curve is self dual, $h=g$ and $H=G$. Therefore if $h+H=g+G$ it must follow that $h=g$ and $H=G$. For any general space curve $H=G=0$. The condition $H=G$ is, then, only necessary for self duality. It follows that the condition $h=g$ is both necessary and sufficient.

For the discussion of conditions (5) to (8) inclusive, the following relations derived by Cayley * will be used:

(a) $t'-t=(r-6)(n-m)$,
(b) $\gamma'-\gamma=(r-8)(m-n)-4(G-H)$,
(c) $q'-q=(r-6)(n-m)+2(G-H)$,
(d) $k'-k=\frac{1}{2}(n-m)(x+y-4r-17)-3(G-H)$.

From the Cayley-Salmon formulas and the limits for a maximum β ‡ it is found that always $r \geq n$ and that for $n \leq 7$, $m \geq n-2$, but for $n \geq 8$, $m \geq n$.

For condition (5), consider relation (a). If $n \neq m$, $r=6$. This condition is satisfied by two curves, duals of each other.

n	m	r	h	H	g	G	α	β	x	y	t	t'
4	6	6	3	0	6	0	4	0	6	4	0	0
6	4	6	6	0	3	0	0	4	4	6	0	0

For condition (6) we find from (b) when $m \neq n$

$$(r-8)=4(G-H)=0.$$

This is satisfied by $r=8$ and $H=G$ in which case $n \leq 8$. These three curves satisfy these conditions:

* A. Cayley, *Collected Papers*, Vol. VIII, p. 77.

‡ T. R. Hollcroft, "Limits for actual double points of space curves," *Bull. Amer. Math. Soc.*, Vol. 31 (1925), pp. 42-55.

n	m	r	h	H	g	G	α	β	x	y	γ	γ'
4	12	8	2	0	38	0	16	0	16	8	0	0
5	9	8	6	0	20	0	8	0	16	12	12	12
6	8	8	8	0	15	0	6	2	15	13	12	12

When $m \neq n$ and $r \neq 8$ the condition

$$(m-n)(r-8)-4(G-H)=0$$

can be satisfied only as a whole. For $r<8$ it is satisfied by the quartic of the first kind for which $H=1$. For $r>8$, curves for which $m \neq n$ can be found to satisfy this condition. In all cases when $r>8$, no general space curve satisfies this condition for $m \neq n$ and only those particular curves for which $r-8$ is a factor of $4(G-H)$.

For condition (7) from (c) results a similar relation

$$(m-n)(r-6)-2(G-H)=0.$$

When $m \neq n$, this may be satisfied by $r=6$ and $G=H$. The same two curve types that satisfy (5), $m \neq n$, also satisfy this condition. When $m \neq n$ and $r \neq 6$, the condition must be satisfied as a whole. As above, only those particular curves for which $r-6$ is a factor of $2(G-H)$ and for which $r>6$ can satisfy it.

For condition (8), from (d) the equation results

$$(n-m)(x+y-4r-17)-6(G-H)=0.$$

When $m \neq n$ it may happen that $x+y=4r-17$ and $G=H$. Eliminating x and y from the relation $x+y=4r-17$ and the Cayley-Salmon formulas for n and m in which it has been assumed that $v=\omega=0$, we obtain

$$m+n=\tfrac{1}{2}(r^2-5r-17).$$

The expression within the parenthesis has an odd value for all values of r. Therefore it is not possible that $x+y=4r-17$ when $v=\omega=0$. There is, then, no general space curve for which $k=k'$ that is not self dual.

If $v \neq 0$, $m+n=\frac{1}{2}(r^2-5r-17-3v)$ and the following curve satisfies this relation: $n=6$, $m=9$, $r=10$, $h=10$, $H=G=0$, $v=1$, $x=30$, $y=27$, $k=k'=282$. Other particular curves may be found similar to this.

When both $\gamma=\gamma'$ and $t=t'$, $n=m$ except for one curve, the quartic of the first kind for which $H=1$.

When $t = t'$ and $q = q'$, the two curve types satisfying (5) for $m \neq n$ satisfy both these conditions.

3. *Singularity limits.* Substituting the relations given in the definition of a self dual space curve in the Cayley-Salmon formulas, there results

$$\begin{aligned} h + H = g + G &= \tfrac{1}{2}(n-1)(n-6) + r - 3p, \\ \alpha = \beta &= 2(n+p-1) - r, \\ x = y &= \tfrac{1}{2}(r-1)(r-6) + n - 3p - \omega, \\ v &= 2(r+p-n-1). \end{aligned}$$

From these formulas we find that if $v = 0$

$$\beta = n + 3p - 3 \quad \text{and} \quad h + H = \tfrac{1}{2}(n^2 - 5n + 8) - 4p.$$

Then since for any space curve

$$h \geq \tfrac{1}{4}n(n-2).^{*}$$

for a self dual space curve when $v = 0$

$$3H + 4\beta \geq \tfrac{1}{4}n(3n-8).$$

When the curve lies on a quadric the equality sign holds for n even and the difference is always less than unity for n odd.

For a curve on a quadric when $v = 0$ and $7 \leq n \leq 15$,

$$[\beta] \leq \tfrac{1}{6}n(n-2).^{\dagger}$$

Substituting this limit for β in the above inequality, we have

$$[H] \geq \tfrac{1}{36}n(n-8).$$

Then on a quadric surface, a self dual space curve for which $v = 0$ and $n > 8$ must have $H > 0$.

In like manner for $n \geq 16$ on a quadric we find

for n even $\qquad [H] \geq \tfrac{1}{6}(n^2 - 15n + 16),$

for n odd $\qquad [H] \geq \tfrac{1}{6}(n^2 - 16n + 19).$

Similar limits may be found for H for self dual space curves for which $v = 0$ on surfaces of the third and higher order.

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* G. Halphen, "Sur quelques propriétés des courbes gauches algébriques," *Bull. de la Soc. de France,* Vol. 2, p. 42.

† T. R. Hollcroft, *loc. cit.,* p. 46.

When $\beta = 0$, from the condition $m = n$ there results

$$v = 2(n + 3p - 3).$$

When also $H = 0$,

$$v = n(3n - 7) - 6h.$$

This is the maximum value of v for a curve of given order and occurs when $m = n$ and $\beta = H = 0$. Then for any space curve without actual point singularities $m \geq n$.

When the curve is on a quadric, the tangent at a linear inflection must be a generator of the quadric. Proper curves that are the intersection of two quadrics can have no linear inflections. The self dual quartic of the first kind has $\beta = 1$, $v = 0$ and the self dual quartic of the second kind has $\beta = 0$, $v = 2$.

Given any general space curve $(\beta = H = v = \omega = 0)$ it may be reduced to a self dual curve of the same order by two methods:

(1) Keep $\beta = H = 0$ and let $v = m - n$. This decreases m, g, x each by v and a by $2v$. This method gives a self dual curve of the same order and genus as the original curve. The same plane projection curve is obtained for all the space curves as v varies from 0 to its maximum value.

(2) Keep $v = 0$ and increase β and H until $m = n$. For $n \leq 8$, β alone may increase until $m = n$ but for $n > 8$, H must also increase. The plane projection curve varies as β and H vary.

A combination of these methods gives a self dual space curve for which neither β nor v vanishes.

For a self dual curve ω may be zero or have values other than zero without affecting the self duality of the curve except that it may cause a lower limit to hold for v.

4. *Plane projection curve when space curve is self dual and space curves that have self dual plane projections.* As a notation for the singularities of a plane curve of order n, class m_1, and genus p, we shall say that it has δ nodes, κ cusps, ι inflections, and τ bitangents.

The plane projection curve of any space curve is of the same order and genus as the space curve and there exist the following relations between the other characteristics of the space curve and those of its plane projection curve:

$$m_1 = r, \quad \iota = m + v, \quad \kappa = \beta, \quad \delta = h + H, \quad \tau = y + \omega.$$

Then for the plane projection curve of a self dual space curve, we have

$$\delta = \tfrac{1}{2}(n-1)(n-6) + m_1 - 3p,$$
$$\kappa = 2(n+p-1) - m_1,$$

with similar expressions for τ and ι respectively with n and m_1 interchanged. But these are entirely general Plücker relations that exist among the characteristics of all plane curves, and therefore, the descriptive properties of the plane projection curve are entirely general. The nodes that are the projections of the apparent double points must lie on an adjoint curve but that is true of any plane curve that is the projection of any space curve. Hence the theorem follows:

The projection of a self dual space curve on a plane is a general plane curve with no additional restrictions due to the fact that the space curve is self dual.

Necessary and sufficient conditions that a plane curve be self dual are

$$\kappa = \iota = n + 2p - 2,$$
$$\delta = \tau = \tfrac{1}{2}(n-2)(n-3) - 3p.$$

The space curve of which a self dual plane curve is the projection has the characteristics

$$n = r,$$
$$h + H = y + \omega = \tfrac{1}{2}(n-2)(n-3) - 3p,$$
$$\beta = m + v = n + 2p - 2.$$

Substituting in this value of $h + H$ the limits for h and p on a quadric, there results

$$4H + 6\beta \leq n(n-2).$$

Since for this curve on a quadric

$$[H] \geq \tfrac{1}{4}(n-2)(n-6) - 3p,$$

$$H \geq 0 \qquad \text{for} \quad [p] \leq \tfrac{1}{12}(n-2)(n-6).$$

and therefore for this value of p the space curve has $[H] \geq 0$ and

$$[\beta] \leq \tfrac{1}{6}n(n-2).$$

Since this is the largest value of β for a curve on a quadric when $v = 0$ and $7 \leq n \leq 15$, then within these limits for n and for $p \leq \tfrac{1}{12}(n-2)(n-6)$, a self dual plane curve can be the projection of a space curve on a quadric. In this case $m = \beta$ when β has its maximum value.

When $\beta = 0$, from the condition $m = n$ there results

$$v = 2(n + 3p - 3).$$

When also $H = 0$,

$$v = n(3n - 7) - 6h.$$

This is the maximum value of v for a curve of given order and occurs when $m = n$ and $\beta = H = 0$. Then for any space curve without actual point singularities $m \geq n$.

When the curve is on a quadric, the tangent at a linear inflection must be a generator of the quadric. Proper curves that are the intersection of two quadrics can have no linear inflections. The self dual quartic of the first kind has $\beta = 1$, $v = 0$ and the self dual quartic of the second kind has $\beta = 0$, $v = 2$.

Given any general space curve ($\beta = H = v = \omega = 0$) it may be reduced to a self dual curve of the same order by two methods:

(1) Keep $\beta = H = 0$ and let $v = m - n$. This decreases m, g, x each by v and α by $2v$. This method gives a self dual curve of the same order and genus as the original curve. The same plane projection curve is obtained for all the space curves as v varies from 0 to its maximum value.

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The plane projection curve of any space curve is of the same order and genus as the space curve and there exist the following relations between the other characteristics of the space curve and those of its plane projection curve:

$$m_1 = r, \quad \iota = m + v, \quad \kappa = \beta, \quad \delta = h + H, \quad \tau = y + \omega.$$

Then for the plane projection curve of a self dual space curve, we have

$$\delta = \tfrac{1}{2}(n-1)(n-6) + m_1 - 3p,$$
$$\kappa = 2(n+p-1) - m_1,$$

with similar expressions for τ and ι respectively with n and m_1 interchanged. But these are entirely general Plücker relations that exist among the characteristics of all plane curves, and therefore, the descriptive properties of the plane projection curve are entirely general. The nodes that are the projections of the apparent double points must lie on an adjoint curve but that is true of any plane curve that is the projection of any space curve. Hence the theorem follows:

The projection of a self dual space curve on a plane is a general plane curve with no additional restrictions due to the fact that the space curve is self dual.

Necessary and sufficient conditions that a plane curve be self dual are

$$\kappa = \iota = n + 2p - 2,$$
$$\delta = \tau = \tfrac{1}{2}(n-2)(n-3) - 3p.$$

The space curve of which a self dual plane curve is the projection has the characteristics

$$n = r,$$
$$h + H = y + \omega = \tfrac{1}{2}(n-2)(n-3) - 3p,$$
$$\beta = m + v = n + 2p - 2.$$

Substituting in this value of $h + H$ the limits for h and p on a quadric, there results

$$4H + 6\beta \leq n(n-2).$$

Since for this curve on a quadric

$$[H] \geq \tfrac{1}{4}(n-2)(n-6) - 3p,$$

$$H \geq 0 \qquad \text{for} \quad [p] \leq \tfrac{1}{12}(n-2)(n-6).$$

and therefore for this value of p the space curve has $[H] \geq 0$ and

$$[\beta] \leq \tfrac{1}{6}n(n-2).$$

Since this is the largest value of β for a curve on a quadric when $v = 0$ and $7 \leq n \leq 15$, then within these limits for n and for $p \leq \tfrac{1}{12}(n-2)(n-6)$, a self dual plane curve can be the projection of a space curve on a quadric. In this case $m = \beta$ when β has its maximum value.

For $n \geqq 16$, the limits for β on a quadric are *

for n even $\qquad [\beta] \leqq \frac{1}{16}(n^2 + 22n - 32)$,

and for n odd $\qquad [\beta] \leqq \frac{1}{16}(n^2 + 24n - 41)$.

The above limit for β when n is even and the limit

$$4H + 6\beta \leqq n(n-2)$$

hold simultaneously for

$$H \geqq \tfrac{1}{32}(5n^2 - 82n + 96)$$

and this value of H can occur for

$$p \leqq \tfrac{1}{32}n(n+6).$$

Similarly, for n odd

$$H \geqq \tfrac{1}{32}(5n^2 - 88n + 123) \quad \text{and}$$
$$p \leqq \tfrac{1}{32}(n-1)(n+9).$$

Then for $n \geqq 16$, a self dual plane curve can be the projection of a space curve on a quadric if n is even for

$$p \leqq \tfrac{1}{32}n(n+6)$$

and if n is odd for

$$p \leqq \tfrac{1}{32}(n-1)(n+9).$$

Similar limits can be found by the same method for surfaces of higher order.

When $m = r$ the plane section of the developable is self dual.

5. *Both the space curve and its plane projection self dual.* It is evident from the preceding section that the necessary condition that both the space curve and its plane projection curve be self dual is that

$$n = m = r.$$

This is also sufficient so far as the characteristics of the two curves are concerned. It is assumed that the projection is made from a general point of space—that is, from any point not on the developable surface of the space curve.

* T. R. Hollcroft, *loc. cit.*, p. 53.

When $n = m = r$, there results:

$$\begin{aligned} h + H = g + G &= \tfrac{1}{2}(n-2)(n-3) - 3p, \\ \alpha = \beta &= n + 2p - 2, \\ x = y &= \tfrac{1}{2}(n-2)(n-3) - 3p - \omega, \\ v &= 2(p-1) = \beta - n. \end{aligned}$$

From these expressions it is evident that such a curve can not be rational and must have the limits

$$p \geq 1, \quad h + H \leq \tfrac{1}{2}n(n-5), \quad \beta \geq n, \quad v \geq 0.$$

If the equality sign holds in any one of these limits, it holds for all.

To find the curve of least order satisfying these conditions, consider $p = 1$, $h = \frac{1}{2}n(n-5)$. Since for any space curve

$$h \geq \tfrac{1}{4}n(n-2)$$

the least n is determined by the inequality

$$2n(n-5) \geq n(n-2)$$

which is satisfied by $n \geq 8$.

A self dual space curve whose plane projection curve is also self dual is therefore possible for $n \geq 8$. For $n = 8$ the curve has the characteristics:

$$n = m = r = 8, \quad p = 1, \quad h = g = 12, \quad \alpha = \beta = 8, \quad v = 0,$$

$$x = y = 12 - \omega, \quad (2, 4).$$

The symbol (a, b) means that the space curve in question is the complete or partial intersection of two surfaces of orders a and b.

Since from any point of the developable, one of the apparent double points of a space curve projects into a cusp of the plane projection curve; from any point of the nodal curve of the developable, two apparent double points project into cusps and form a triple point of the nodal curve, three, certain self dual space curves for $n < 8$ may project into self dual plane curves from points on the developable. The following results hold for any n:

A self dual space curve for which

$$\beta = n + 2p - 3, \quad h + H = \tfrac{1}{2}n(n-5) - 3p + 4, \quad r = n + 1,$$

projects into a self dual plane curve from any point of the developable;

4

$$\beta = n + 2p - 4, \quad h + H = \tfrac{1}{2}n(n-5) - 3p + 5, \quad r = n + 2, \quad n \geq 3,$$

projects into a self dual plane curve from any point of the nodal curve;

$$\beta = n + 2p - 5, \quad h + H = \tfrac{1}{2}n(n-5) - 3p + 6, \quad r = n + 3, \quad n \geq 8,$$

projects into a self dual plane curve from a triple point of the nodal curve.

For $n < 8$, these curves are all rational.

The limit $\gamma \geq 0$ gives the following limit for β,

$$\beta \leq \tfrac{1}{6}[r(n-6) + 4(n - \omega - H) - 2v].$$

When $r = n$, $\omega = 0$, $v = \beta - n$, this becomes

$$\beta \leq \tfrac{1}{8}(n^2 - 4H).$$

When $n = m = r$, this serves as an upper bound to β for any surface.

When $H = 0$, $\beta \leq \tfrac{1}{8}n^2$ and since also $\beta = n + 2p - 2$, we find for p the limit

$$p \leq \tfrac{1}{16}(n-4)^2.$$

When $H > 0$, consider curves on quadrics, cubics, etc. For curves on quadrics, since $h \geq \tfrac{1}{4}n(n-2)$, when $n = m = r$

$$H \leq \tfrac{1}{4}(n-2)(n-6) - 3p.$$

Substituting this value of H in the limit $\beta \leq \tfrac{1}{8}(n^2 - 4H)$, we obtain

$$\beta \leq \tfrac{1}{2}(2n + 3p - 3).$$

But $\beta = n + 2p - 2$. These two values of β are satisfied simultaneously by $p \leq 1$. Since p can not be zero, there remains only $p = 1$. Therefore curves for which $n = m = r$ can exist as complete intersections (or for n odd with a straight line residual) on quadric surfaces only for the genus unity.

For $n \geq 16$, the maximum number of cusps of a space curve on a quadric for n even and $H \geq 0$ is

$$[\beta] \leq \tfrac{1}{16}(n^2 + 22n - 32 - 8H).$$

Substituting in this the value of H obtained above, there results

$$\beta \leq \tfrac{1}{16}(38n - n^2 - 56 + 24p).$$

Solving this with $\beta = n + 2p - 2$, we obtain

$$p \leq \tfrac{1}{8}(22n - n^2 - 24)$$

which gives $p \geq 1$ for $n \leq 20$.

Similarly, for n odd we find that $p \geq 1$ for $n \leq 21$.

Combining this result with that obtained from the limit $\gamma \geq 0$, we have finally:

Self dual space curves whose plane intersections are also self dual curves can exist as complete intersections (or for n odd with no more than a straight line residual) on quadric surfaces only for an order $n \leq 21$ and the genus unity.

In a similar way for curves on cubic surfaces such that the residual does not exceed the order two, we find that space curves for which $n = m = r$ may exist for $n = 23$ and $n = 26$ and for the following genus limits:

$$n \leq 14, \quad 1 \leq p \leq \tfrac{1}{12}(n^2 - 6n + 20);$$

$$15 \leq n \leq 23$$

$$1 \leq p \leq \tfrac{1}{9}(24n - n^2 - 27) \quad \text{for } (\tfrac{1}{3}n, 3);$$

$$1 \leq p \leq \tfrac{1}{9}(28n - n^2 - 43) \quad \text{for } (\tfrac{1}{3}[n+1], 3);$$

$$1 \leq p \leq \tfrac{1}{9}(26n - n^2 - 34) \quad \text{for } (\tfrac{1}{3}[n+2], 3).$$

For $n = 26$, the curve lies on $(3, 9)$, $p = 1$, $\beta = 26$, $h = 200$, $H = 73$, $r = m = 26$, $v = 0$.

Similar limits can be found for curves on quartic surfaces and surfaces of higher order.

From the fact that $\gamma \geq 0$, we have found that for $n = m = r$ when $H = 0$

$$\beta \leqq \tfrac{1}{8}n^2 \quad \text{and} \quad p \leq \tfrac{1}{16}(n-4)^2.$$

This serves as an effective limit on surfaces of orders 2 and 3 whether the curve be a complete intersection or not, but for surfaces of higher order it is only an upper bound. This does not mean, however, that all such curves on quadric and cubic surfaces within the genus given exist. This will be illustrated in the following examples.

The curve of lowest order for which $n = m = r$ on a cubic surface is of order 9. This curve has $p = 1$, $h + H = 18$, $\beta = 9$, and may be considered on the surfaces $(3, 3)$, $h = 18$, $H = 0$ or $(2, 5)$, $h = 16$, $H = 2$.

For $n = 10$, $p = 1$, $h + H = 25$, $\beta = 10$. This curve may be considered as $(2, 5)$, $h = 20$, $H = 5$ or $(3, 4)$, $h = 24$, $H = 1$ or $(4, 4)$, $h = 25$. Also for $n = 10$, $p = 2$, $(2, 6)$, $h = 21$, $H = 1$, $\beta = 12$, $v = 2$.

For $n = 11$, curves of genus 1, 2 and 3 as given by the above limit are all found to exist.

For $n \leq 11$, all the curves within the genus given by the above limits have been found to exist. For $n \geq 12$, however, for each order some of the curves within these genus limits do not exist.

For $n = 12$, the highest genus 4 given by the above limit does not belong to a curve that exists, since for $n = 12$, $h \neq 33$. The curve of highest genus that exists has $p = 3$, $\beta = 16$, $v = 4$, $(2, 8)$, $h = 34$, $H = 2$ or $(3, 4)$, $h = 36$, $H = 0$.

For $n = 13$, $p = 5$, the curve does not exist. For $p = 4$, $\beta = 19$, $v = 6$, $(2, 9)$, $h = 42$, $H = 1$. For $n = 14$, the curve does not exist for $p = 6$, but for $p = 5$, $\beta = 22$, $(2, 10)$, $h = 51$, $H = 0$. When $n = 15$, curves for $p = 6, 7$ do not exist and for $p = 5$, $(4, 4)$, $\beta = 23$, $h = 63$, $H = 0$. Etc.

6. *Existence of curves.* In the preceding section, the term "exist" has been used to mean that certain curves of given order, or given surfaces and having no actual singularities exist. As to whether such a space curve still exists when actual singularities within the limits prescribed are added to its singularities, the same discussion holds as that given in the last section of the paper, "Limits for actual double points on space curves." *

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* T. R. Hollcroft, *loc. cit.*, pp 54-55.

Expansions in Terms of Certain Polynomials Connected with the Gamma-Function.

BY BORDEN PARKER HOOVER.

I. INTRODUCTION.

1. *Statement of the Problem.* The nth difference of any solution of the equation $f(x+1) = xf(x)$ can be written as the product of $f(x)$ and a polynomial of degree n $(n = 1, 2, 3, \cdots)$. We denote this polynomial by $P_n(x)$ and define $P_0(x)$ to be 1.

Let $\nabla u(x)$ denote the sum, $u(x+1) + u(x)$. Perform n times on $f(x)$ the operation indicated by ∇. The result can be written as the product of $f(x)$ and a polynomial of degree n $(n = 1, 2, 3, \cdots)$. This polynomial we denote by $\mathfrak{P}_n(x)$ and define $\mathfrak{P}_0(x)$ to be 1.

The polynomials so defined are:

$$(1) \qquad \begin{aligned} P_n(x) &= x(x+1)\cdots(x+n-1) - nx(x+1)\cdots(x+n-2) \\ &\quad + \tbinom{n}{2}x(x+1)\cdots(x+n-3) - \cdots + (-1)^n, \end{aligned}$$

$$(2) \qquad \begin{aligned} \mathfrak{P}_n(x) &= x(x+1)\cdots(x+n-1) + nx(x+1)\cdots(x+n-2) \\ &\quad + \tbinom{n}{2}x(x+1)\cdots(x+n-3) + \cdots + 1. \end{aligned}$$

It is the purpose of this paper to consider some of the properties of these two sets of polynomials; in particular, to obtain expansions of arbitrary functions in terms of these polynomials and functions associated with them; and, finally, to generalize the expansion theory to functions of any finite number of variables. The method of Neumann * will be used in obtaining the expansions.

II. PROPERTIES OF THE POLYNOMIALS.

2. *Recurrence Formulae for the Polynomials.* The polynomials $P_n(x)$ satisfy the following relations:

$$\text{(A)} \quad P_{n+2}(x) - (x+n)P_{n+1}(x) - (n+1)P_n(x) = 0,$$

$$\text{(B)} \quad (x+1)P_n(x+2) - (x+n+2)P_n(x+1) + P_n(x) = 0,$$

$$\text{(C)} \quad xP_{n+1}(x+1) - (x+n+1)P_{n+1}(x) - (n+1)P_n(x) = 0,$$

$$\text{(D)} \quad P_{n+1}(x) - xP_n(x+1) + P_n(x) = 0.$$

* See Watson's *Theory of Bessel Functions* (London, 1922), Chapter 9.

To obtain (A) make use of the relation $f(x+1) = xf(x)$, whence $\nabla f(x) = f(x+1) - f(x) = (x-1)f(x)$; find the successive differences of $f(x)$, making use of the identity, $\nabla\{u(x)v(x)\} = u(x+1)\cdot\nabla v(x) + \nabla u(x)\cdot v(x)$, in the right member; in the nth difference so obtained replace n by $n+2$; then make use of the definition of $P_n(x)$. To get (B) write equations giving the nth differences of $f(x)$ and $f(x+1)$ obtained as above; from the first of these equations get two new equations by replacing x by $x+1$ and $x+2$, respectively; from the second get three new equations by putting (1) $x = x+1$, (2) $n = n-1$, and (3) $x = x+1$ and $n = n-1$; by elimination, one obtains from these seven equations a relation in the nth differences of $f(x)$, $f(x+1)$, and $f(x+2)$; then make use of the definition of $P_n(x)$. (C) comes at once from the expression for the nth difference of $f(x)$ on replacing n by $n+1$ and dividing out the factor $f(x)$. Finally, subtract (C) from (A) and replace n by $n-1$ to get (D).

These formulae will be useful for obtaining explicit formulae for $P_n(x)$.

Corresponding recurrence relations for the $\mathfrak{P}_n(x)$, obtained in a similar manner, are:

$$
\begin{aligned}
&(A)\quad \mathfrak{P}_{n+2}(x) - (x+n+2)\,\mathfrak{P}_{n+1}(x) + (n+1)\mathfrak{P}_n(x) = 0,\\
&(B)\quad (x+1)\,\mathfrak{P}_n(x+2) - (x+n)\mathfrak{P}_n(x+1) - \mathfrak{P}_n(x) = 0,\\
&(C)\quad \mathfrak{P}_{n+1}(x+1) - (x+n+1)\,\mathfrak{P}_n(x+1) - \mathfrak{P}_n(x) = 0,\\
&(D)\quad \mathfrak{P}_{n+1}(x+1) - (n+1)\,\mathfrak{P}_n(x+1) - \mathfrak{P}_{n+1}(x) = 0.
\end{aligned}
$$

3. *Explicit Formulae for the Polynomials.* The Gamma-function is a solution of the equation $f(x+1) = xf(x)$. If $R(x)$ denotes the real part of x, $\Gamma(x)$ can be written in the form,

$$\Gamma(x) = \int_0^\infty v^{x-1}\,e^{-v}\,dv, \qquad R(x) > 0.$$

On finding the nth difference of $\Gamma(x)$ from this integral representation and remembering that $P_n(x)\cdot\Gamma(x) = \nabla^n\Gamma(x)$, we are led to the relation,

$$(2)\qquad P_n(x) = 1/\Gamma(x)\int_0^\infty v^{x-1}\,e^{-v}(v-1)^n dv, \qquad R(x) > 0.$$

A contour integral representation of $P_n(x)$ can be obtained by using the Laplace transformation in connection with the equation,

$$(x+1)u(x+2) - (x+n+2)u(x+1) + u(x) = 0,$$

which we know from (B) to be satisfied by $P_n(x)$. We seek to determine a

function $\phi(t)$ and a path of integration so that we can express $u(x)$ in the form,

$$u(x) = \int t^{x-1}\phi(t)\,dt.$$

Substituting this assumed integral for $u(x)$ in the foregoing difference equation and proceeding formally, we are led to the formal solution below, which is easily shown, by substitution, to satisfy the given difference equation:

$$u(x) = \int^{(0+)} \frac{(t+1)^{x+n-1}\, e^{-(t/1+t)}\, dt}{t^{n+1}},$$

where the contour is any closed curve including the point zero and excluding the point -1. Now let $s(t+1) = -t$. This transformation clearly leaves the character of the contour unchanged and we have the relation,

$$u(x) = (-1)^n \int^{(0+)} \frac{(1+s)^{-x}\, e^{s}}{s^{n+1}}\, ds.$$

It is easy to evaluate this last written integral by making use of the theory of residues. Comparing the value of $u(x)$ so obtained with the expression for $P_n(x)$ given by (1), we identify $P_n(x)$ as $(n!/2\pi i)u(x)$. Hence we can write the two formulae for $P_n(x)$,

$$\text{(3)} \qquad P_n(x) = n!/2\pi i \int^{(0+)} \frac{(t+1)^{x+n-1}\, e^{-(t/1+t)}}{t^{n+1}}\, dt,$$

$$\text{(4)} \qquad P_n(x) = n!(-1)^n/2\pi i \int^{(0+)} \frac{(1+s)^{-x}\, e^{s}}{s^{n+1}}\, ds,$$

valid for all values of x.

The following corresponding formulae for $\mathfrak{P}_n(x)$ may be similarly derived:

$$\text{(2')} \qquad \mathfrak{P}_n(x) = 1/\Gamma(x) \int_0^\infty v^{x-1}\, e^{-v}(v+1)^n dv, \qquad R(x) > 0,$$

$$\text{(3')} \qquad \mathfrak{P}_n(x) = n!/2\pi i \int^{(0+)} \frac{(1+t)^{x+n-1}\, e^{\ (t/1+t)}\, dt}{t^{n+1}},$$

$$\text{(4')} \qquad \mathfrak{P}_n(x) = (-1)^n n!/2\pi i \int^{(0+)} \frac{(1+s)^{-x}\, e^{-s}}{s^{n+1}}\, ds.$$

4. *Generating Functions for the Polynomials.* The formulae (3) and (4) may be used to define the $P_n(x)$ as coefficients in infinite series. Noticing that $P_n(x)$ is the nth coefficient in the Maclaurin expansion of the inte-

grand except for a factor independent of the integration variable, we have at once the equations,

$$(5)\qquad (t+1)^{x+n-1}\, e^{-(t/1+t)} = \sum_{n=0}^{\infty} P_n(x)\,(t^n/n!),$$

$$(6)\qquad (1+s)^{-x}\, e^{s} = \sum_{n=0}^{\infty} (-1)^n P_n(x)\,(s^n/n!).$$

Making use of (3′) and (4′) we get similar generating functions for the $\mathfrak{P}_n(x)$. They are:

$$(5')\qquad (t+1)^{x+n-1}\, e^{(t/1+t)} = \sum_{n=0}^{\infty} \mathfrak{P}_n(x)\,(t^n/n!),$$

$$(6')\qquad (1+s)^{-x}\, e^{-s} = \sum_{n=0}^{\infty} (-1)^n \mathfrak{P}_n(x)\,(s^n/n!).$$

III. Expansions of Arbitrary Functions in Terms of the Polynomials and the Associated Functions.

5. *Expansion of $1/(t-x)$ in a Series of Binomial Coefficients.* By a method used by Bendixson,* we are led to the identity,

$$1/(t-x) = 1/t + \frac{x}{t(t+1)} + \frac{x(x+1)}{t(t+1)(t+2)} + \cdots$$
$$+ \frac{x(x+1)(x+2)\cdots(x+n-1)}{t(t+1)(t+2)\cdots(t+n)} + R_n(t,x),$$

where

$$R_n(t,x) = \frac{x(x+1)\cdots(x+n)}{t(t+1)\cdots(t+n)} \quad \frac{1}{t-x}$$
$$= \frac{\Gamma(x+n+1)\cdot\Gamma(t)}{\Gamma(t+n+1)\cdot\Gamma(x)} \cdot \frac{1}{t-x}\ .$$

Making use of the asymptotic character of the Gamma-function, it is clear that the remainder term approaches zero as n increases indefinitely provided that $R(t) > R(x)$. Hence we have the relation,

$$(7)\qquad 1/(t-x) = 1/t + \sum_{n=1}^{\infty} \frac{x(x+1)\cdots(x+n-1)}{t(t+1)\cdots(t+n)}, \quad R(t) > R(x).$$

6. *The Binomial Coefficients in Terms of the Polynomials $P_n(x)$ and $\mathfrak{P}_n(x)$.* If the $n+1$ equations, obtained by lettng n take the values $0, 1, 2, \cdots, n$ in (1) in order, are written and then multiplied in reverse order

* *Acta Mathematica*, Vol. 9, pp. 7, 8.

by the binomial coefficients for the exponent n, and added, we get the equation,

$$x(x+1)\cdots(x+n-1) = \sum_{k=0}^{n} \binom{n}{k} P_k(x), \tag{8}$$

and in a similar way,

$$x(x+1)\cdots(x+n-1) = \sum_{k=0}^{n} (-1)^{n+k} \binom{n}{k} \mathfrak{P}_k(x). \tag{8'}$$

7. *A Modified Form of the Cauchy Integral Theorem.* We shall make use of the lemma,

LEMMA. *Let $f(x)$ be analytic, except, perhaps, at infinity, on and in the region to the left [right] of the line $R(x) = a$. Furthermore, let $f(x)$ have the property that, for all values of x in this domain, there exist the positive numbers ϵ and K, such that we may write the inequality,*

$$| x^{\epsilon} f(x) | < K.$$

Then $f(x)$ may be written in the form, when $R(x) < a[> a]$,

$$f(x) = 1/2\pi i \int_{a-i\infty}^{a+i\infty} \frac{f(t)\,dt}{t-x} \left[1/2\pi i \int_{a-i\infty}^{a+i\infty} \frac{f(t)\,dt}{x-t} \right]. \tag{9}$$

The relation just written is obtained as a limiting form of the ordinary statement of Cauchy's Integral Theorem for x within each of a sequence of semicircles C_k, $(k = 1, 2, \cdots)$, having the properties: the center of C_k of radius ρ_k is at a fixed point P on the line $R(x) = a$, and its diameter is on the same line; C_k encloses only points to the left [right] of $R(x) = a$; $\rho_{k-1} < \rho_k < \rho_{k+1}$ and ρ_k approaches infinity as k becomes infinite. The restriction on $f(x)$ is such that the part of the integral contributed by integrating along the arc of C_k approaches zero as k becomes infinite.

8. *Formal Expansion of $1/(t-x)$ in Terms of the Polynomials $P_n(x)$ and $\mathfrak{P}_n(x)$.* Using the results given by (7) and (8) we have at once the expression for $1/(t-x)$,

$$\begin{aligned} 1/(t-x) = {} & \frac{P_0(x)}{t} + \frac{P_0(x) + P_1(x)}{t(t+1)} \\ & + \frac{P_0(x) + 2P_1(x) + P_2(x)}{t(t+1)(t+2)} + \cdots \\ & + \frac{P_0(x) + \binom{n}{1} P_1(x) + \cdots + P_n(x)}{t(t+1)\cdots(t+n)} + \cdots, \quad R(t) > R(x). \end{aligned} \tag{10}$$

Assuming temporarily the validity of any sort of rearrangement, we reckon out the coefficients of the successive polynomials and get the relation,

$$(11) \qquad 1/(t-x) = \sum_{n=0}^{\infty} P_n(x)\, Q_n(t),$$

where

$$Q_n(t) = \frac{\binom{n}{n}}{t(t+1)\cdots(t+n)} + \frac{\binom{n+1}{n}}{t(t+1)\cdots(t+n+1)} + \cdots.$$

Proceeding in a like manner, we have a corresponding formally derived series expressing $1/(t-x)$ in terms of the $\mathfrak{P}_n(x)$,

$$(11') \qquad 1/(t-x) = \sum_{n=0}^{\infty} \mathfrak{P}_n(x)\, \mathfrak{Q}_n(t),$$

where

$$\mathfrak{Q}_n(t) = \frac{\binom{n}{n}}{t(t+1)\cdots(t+n)} - \frac{\binom{n+1}{n}}{t(t+1)\cdots(t+n+1)} + \cdots.$$

The functions $Q_n(t)$ and $\mathfrak{Q}_n(t)$ will hereafter be referred to as the functions associated with $P_n(x)$ and $\mathfrak{P}_n(x)$, respectively, in the formally obtained series given above. We proceed to the question of convergence of the foregoing series.

9. *Dominating Functions for the Polynomials $P_n(x)$ and $Q_n(t)$.*

Case I. When $R(x) = r > 0$, we may write, using (2),

$$\Gamma(x)P_n(x) = \int_0^1 v^{x-1} e^{-v}(v-1)^n\, dv + \int_1^{\infty} v^{x-1} e^{-v}(v-1)^n\, dv.$$

Whence we may write the inequality,

$$|\Gamma(x)P_n(x)| \leqq \int_0^1 v^{r-1} e^{-v} |v-1|^n\, dv - \int_0^1 v^{r-1} e^{-v} v^n\, dv$$
$$+ \int_0^{\infty} v^{r+n-1} e^{-v}\, dv \leqq 1/r + \Gamma(r+n).$$

Therefore

$$(12) \qquad |P_n(x)| \leqq \frac{1}{|\Gamma(x)|}\{1/r + \Gamma(r+n)\} \leqq \Gamma(r+n)g(x),$$

where $g(x)$ is independent of n.

Case II. When $R(x) = -a \leqq 0$, we get from (4) of section 3, by taking the contour to be a circle with center at the point zero and with radius $\rho < 1$ and putting $x = -a + ib$,

$$P_n(x) = \frac{(-1)^n n!}{2\pi i} \int_0^{2\pi} \frac{(1+\rho\cos\theta + i\rho\sin\theta)^{a+ib}\cdot\exp(\rho\cos\theta + i\rho\sin\theta)}{\rho^n \exp(ni\theta)}\, d\theta.$$

Hence we may write the inequality,

$$\begin{aligned} |P_n(x)| &\leqq \frac{n!e^\rho}{2\rho^n\pi}\int_0^{2\pi} \exp\Big(a\cdot\log\sqrt{1+2\rho\cos\theta+\rho^2} \\ &\qquad + b\cdot\arctan\frac{\rho\sin\theta}{1+\rho\cos\theta}\Big)\, d\theta \\ &\leqq e^\rho n!/\rho^n \exp\{a\cdot\log(1+\rho) + |b|\pi/2\}, \qquad (\rho<1), \\ &\leqq \lim_{\rho\to 1} n!/\rho^n \exp\{\rho + a\cdot\log(1+\rho) + |b|\pi/2\} \\ &= n!\exp(1 + a\cdot\log 2 + |b|\pi/2). \end{aligned}$$

Hence

(13) $$|P_n(x)| \leqq c(x)\, n!,$$

where $c(x)$ is a quantity independent of n.

It is convenient to find a dominating function for $Q_n(t)$ by expressing it as a definite integral. Using the formal relation (11) and obtaining two more by replacing x by $x+1$, and $x+2$, respectively, and then multiplying them in order by 1, $-(x+2)$, and $(x+1)$, and adding, we get, after making use of (B),

(14) $$\frac{1}{t-x} - \frac{x+2}{t-x-1} + \frac{x+1}{t-x-2} = \sum_{n=0}^{\infty} nP_n(x+1)Q_n(t).$$

But the left member of this equation is equal to

$$\frac{1}{t-x} - \frac{t+1}{t-x-1} + \frac{t-1}{t-x-2}.$$

Hence, if we again use (11) and put $t=t+1$, and $t-1$, respectively, then multiply in order by $-(t+1)$, 1, and $t-1$, and add, the right member of the resulting equation should be equivalent to the right member of (14). By equating coefficients of corresponding terms and replacing t by $t+1$ in the equation so obtained we are led to the recurrence relation,

(B′) $$Q_n(t+2) - (t+n+2)Q_n(t+1) + tQ_n(t) = 0.$$

Laplace's transformation in connection with this equation leads at once to the integral below, which is easily verified by substitution to be a solution of (B′),

$$\int_0^1 (1-u)^{t-1} u^n e^u\, du, \qquad R(t) > 0.$$

This integral can be identified as $n!\,Q_n(t)$ if, after expanding the exponential in the integrand and integrating term by term, it is compared with the series for $Q_n(t)$ in section 8. Using the form for $Q_n(t)$ so obtained and integrating by parts, we have $Q_n(t)$ in the form,

$$(15)\qquad Q_n(t) = 1/tn!\int_0^1 (1-u)^t u^{n-1} e^u (u+n)\,du,\quad R(t) > 0.$$

Denoting $R(t)$ by s, we have the inequality,

$$(16)\qquad |\,Q_n(t)\,| \leqq \frac{e(n+1)}{|\,t\,|\,n}\cdot\frac{\Gamma(s+1)}{\Gamma(s+n+1)}\,.$$

10. *Convergence Proof for the Series Formally Representing* $1/(t-x)$. It is easy to show that in the series obtained by replacing the terms of the series in the right member of (11) by the functions found in the preceding section which dominate their absolute values, the ratio of the nth and the $(n+1)$th terms can be written in the form,

$$1 + \frac{s-r+1}{n} + O(1/n).$$

Therefore, the series in the right member of (11) converges absolutely, provided $s > r$.

In a similar way the absolute convergence of the series in the right member of (11′) can be established.

11. *Equivalence of* $1/(t-x)$ *and its Formal Expansions in Terms of* $P_n(x)$ *and* $\mathfrak{P}_n(x)$. Consider the coefficients $Q_n(t)$ of $P_n(x)$ in the right member of (11) as being written at length in their infinite series form. Then consider the double series formed by arranging the terms so that the general term has the form,

$$\begin{aligned} u_{m,n} &= P_m(x)\,\frac{\binom{n}{m}}{t(t+1)\cdots(t+n)}\,, && n \geqq m,\\ &= 0, && n < m. \end{aligned}$$

If we let $\{S_{m,n}\}$ denote the double sequence of positive numbers obtained by letting m and n vary independently where $S_{m,n}$ represents the sum of the absolute values of the terms $u_{m,n}$ in the rectangle composed of the first n columns of the first m rows, then the sequence $\{S_{m,n}\}$ is monotone. Furthermore, since

$$\left|\frac{1}{t(t+1)\cdots(t+n)}\right| \leqq \frac{1}{s(s+1)\cdots(s+n)}\,,\quad s = R(t) > 0,$$

it is easy to establish the existence of the limit,

$$\lim_{m\to\infty}\ \lim_{n\to\infty}\ S_{m,n},$$

since the same dominating functions already used for the $Q_n(t)$ apply here also. We conclude then to the existence of the sum of the double series whose general term is $|\,u_{m,n}\,|$.* Hence, the double series whose general term is $u_{m,n}$ converges absolutely whence the sum by rows equals the sum by columns. But we have already seen the equivalence of $1/(t-x)$ and the sum by columns in section 9, so that we have established the validity of the expansion given by (11) provided that $R(t) > 0$ and $R(t) > R(x)$.

In an analogous manner the validity of (11′) can be established.

12. *Recurrence Formulae for the Associated Functions* $Q_n(t)$ and $\mathfrak{Q}_n(t)$. In addition to (B′) obtained in section 9, the $Q_n(t)$ satisfy the following relations:

$$\begin{aligned}
&(\mathrm{A}') \quad (n+2)Q_{n+2}(t)+(t+n)Q_{n+1}(t)-Q_n(t)=0,\\
&(\mathrm{C}') \quad tQ_{n+1}(t)-Q_{n+1}(t+1)-Q_n(t+1)=0,\\
&(\mathrm{D}') \quad Q_{n+1}(t+1)-(t+n+1)Q_{n+1}(t)+Q_n(t)=0.
\end{aligned}$$

These formulae can be verified by using the integral expression for $Q_n(t)$ which is valid for $R(t) > 0$. It follows that the formulae hold wherever the functions $Q_n(t)$ are defined by analytic continuation.

Analogous formulae may be written for the functions $\mathfrak{Q}_n(t)$ associated with the polynomials $\mathfrak{P}_n(x)$. They are:

$$\begin{aligned}
&(\boldsymbol{A}') \quad (n+2)\,\mathfrak{Q}_{n+2}(t)-(t+n+2)\,\mathfrak{Q}_{n+1}(t)+\mathfrak{Q}_n(t)=0,\\
&(\boldsymbol{B}') \quad \mathfrak{Q}_n(t+2)+(t+n)\mathfrak{Q}_n(t+1)-t\,\mathfrak{Q}_n(t)=0,\\
&(\boldsymbol{C}') \quad t\mathfrak{Q}_{n+1}(t)+\mathfrak{Q}_{n+1}(t+1)-\mathfrak{Q}_n(t+1)=0,\\
&(\boldsymbol{D}') \quad \mathfrak{Q}_{n+1}(t+1)+(t+n-1)\,\mathfrak{Q}_{n+1}(t)+\mathfrak{Q}_n(t)=0.
\end{aligned}$$

13. *Expansions of Functions in Terms of* $P_n(x)$ *and* $\mathfrak{P}_n(x)$. We have seen that a function $f(x)$ may be expressed in the form (9) provided the restrictions imposed there are satisfied. Using the results obtained there and substituting for $1/(t-x)$ its expansion (11) in terms of the $P_n(x)$, the validity of which was established in the preceding section, we have the relation,

$$f(x)=1/2\pi i\int_{a-i\infty}^{a+i\infty} f(t)\,\{\sum_{n=0}^{\infty} P_n(x)Q_n(t)\}dt.$$

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* See, *e. g.*, a theorem on page 465 in Hobson's *Theory of Functions of a Real Variable*, First Edition.

To justify integration term by term in the right member, we first show that the integrals of the separate terms exist. We have

$$\left| \int_{a-i\infty}^{a+i\infty} Q_n(t)f(t)\,dt \right| = \left| \int_{-\infty}^{+\infty} Q_n(a+is)f(a+is)\,ds \right|$$
$$= \left| \int_{-\infty}^{+\infty} \frac{Q_n(a+is)(a+is)^\epsilon f(a+is)\,ds}{(a+is)^\epsilon} \right|$$
$$\leqq \frac{e(n+1)\,\Gamma(a+1)K}{n\,\Gamma(a+n+1)} \int_{-\infty}^{+\infty} \frac{ds}{(a^2+s^2)^{(1+\epsilon)/2}}$$

where account has been taken of the restriction on $f(x)$ and the bounds already obtained for the polynomials and the associated series. Hence, the above inequality holds provided that a is a positive number. It is obvious that the integral in the right member of the inequality exists and so we may integrate term by term if $R_n(x,t)$ approaches zero as n becomes infinite, where

$$R_n(x,t) = \int_{a-i\infty}^{a+i\infty} \sum_{k=n+1}^{\infty} \{P_k(x)\cdot Q_k(t)\}\cdot f(t)\,dt.$$

It is clear from (11) and (16) that the series

$$\sum_{n=0}^{\infty} P_n(x)tQ_n(t)$$

converges for all values of t on the line $R(t)=a$ and, furthermore, that this convergence is uniform with regard to t. Hence, we have the inequality,

$$|R_n(x,t)| \leqq \int_{a-i\infty}^{a+i\infty} 1/|t|\ \delta_n(x)\ |f(t)|\ dt = \int_{a-i\infty}^{a+i\infty} \frac{|t^\epsilon f(t)\ |\ \delta_n(x)\ |\ dt|}{|t|^{1+\epsilon}},$$

where $\delta_n(x)$ is a function independent of t and has the property that it approaches zero as n becomes infinite. Therefore, using the hypothesis made on $f(x)$, we have

$$\lim_{n\to\infty} |R_n(x,t)| \leqq \lim_{n\to\infty} \delta_n(x)K \int_{-\infty}^{+\infty} \frac{ds}{(a^2+s^2)^{(1+\epsilon)/2}} = 0.$$

These results may be summed up in the following theorem:

THEOREM I. *Let $f(x)$ be analytic, except, perhaps, at infinity, for x on or to the left of the line $R(x)=a$, $a>0$. Furthermore, let $f(x)$ have the property that for all values of x in this domain there exist positive numbers ϵ and K such that we have the inequality,*

$$|x^\epsilon f(x)| < K.$$

Then $f(x)$ has the expansion, when $R(x) < a$,

$$f(x) = \sum_{n=0}^{\infty} a_n P_n(x),$$

where

$$a_n = 1/2\pi i \int_{a-i\infty}^{a+i\infty} Q_n(t) f(t)\,dt.$$

A similar expansion in terms of the polynomials $\mathfrak{P}_n(x)$ is obtained in an analogous manner. This gives the theorem:

THEOREM II. *Let $f(x)$ be analytic, except, perhaps, at infinity, for x on or to the left of the line $R(x) = a$, $a > 0$. Furthermore, let $f(x)$ have the property that for all values of x in this domain there exist positive numbers ϵ and K such that we have the inequality,*

$$| x^{\epsilon} f(x) | < K.$$

Then $f(x)$ has the expansion, for $R(x) < a$,

$$f(x) = \sum_{n=0}^{\infty} a_n \mathfrak{P}_n(x),$$

where

$$a_n = 1/2\pi i \int_{a-i\infty}^{a+i\infty} \mathfrak{Q}_n(t) f(t)\,dt.$$

14. *Expansions in Terms of the Associated Functions $Q_n(t)$ and $\mathfrak{Q}_n(t)$.* If the roles of t and x are interchanged in (11) we have the valid expansion,

$$1/(x - t) = \sum_{n=0}^{\infty} Q_n(x) \cdot P_n(t), \quad R(x) > R(t).$$

Substituting this series for $1/(x - t)$ in equation (9) of section 7 we have the relation,

$$f(x) = 1/2\pi i \int_{a-i\infty}^{a+i\infty} \sum_{n=0}^{\infty} Q_n(x) \cdot P_n(t) f(t)\,dt.$$

To enable us to integrate term by term in the right member we impose a further restriction on $f(x)$ than imposed in (9). Let $f(x)$ be such that for all values of x on the line $R(x) = a$, $a > 0$, we have

$$| x^{a+\frac{1}{2}+\epsilon} e^{|x| \pi/2} f(x) | < K.$$

Making use again of the asymptotic character of the Gamma-function in con-

nection with the bound given in (12), we can use methods similar to those used in establishing the previous theorems and have the theorem:

THEOREM III. *Let $f(x)$ be analytic, except, perhaps, at infinity, for x on and to the right of the line $R(x)=a$, $a>0$. Furthermore, let $f(x)$ be such that, for x in this domain, there exist positive numbers ϵ and K such that we have the inequality,*

$$| x^{\epsilon} f(x) | < K;$$

and for x on the line $R(x)=a$, we have the additional inequality,

$$| x^{a+\frac{1}{2}+\epsilon} e^{|x| \pi/2} f(x) | < K.$$

Then $f(x)$ has the expansion, when $R(x)>a$,

$$f(x) = \sum_{n=0}^{\infty} b_n Q_n(x),$$

where

$$b_n = 1/2\pi i \int_{a-i\infty}^{a+i\infty} P_n(t) f(t)\, dt.$$

The analogous theorem giving an expansion of $f(x)$ in terms of the $\mathcal{Q}_n(x)$ is stated below:

THEOREM IV. *Let $f(x)$ be analytic, except, perhaps, at infinity, for x on and to the right of the line $R(x)=a$, $a>0$. Furthermore, let $f(x)$ have the property that, for all values of x in this domain there exist positive numbers ϵ and K such that we have the inequality,*

$$| x^{\epsilon} f(x) | < K;$$

and for x on the line $R(x)=a$, we have the additional inequality,

$$| x^{a+\frac{1}{2}+\epsilon} e^{|x| \pi/2} f(x) | < K.$$

Then $f(x)$ has the expansion, when $R(x)>a$,

$$f(x) = \sum_{n=0}^{\infty} \beta_n \mathcal{Q}_n(x),$$

where

$$\beta_n = 1/2\pi i \int_{a-i\infty}^{a+i\infty} \mathcal{P}_n(t) f(t)\, dt.$$

15. *Expansion Valid in a Strip Parallel to the Imaginary Axis.* Let a and b be two positive numbers, $b>a$. Let $f(x)$ be such that for all values

of x on and between the lines $R(x) = a$ and $R(x) = b$ there exist positive numbers ϵ and K such that we have the inequality,

$$|\, x^{\epsilon} f(x) \,| < K.$$

It is a straightforward process to show that Cauchy's integral theorem can be modified, when x is on the interior of this strip and $f(x)$ is restricted as stated above, so that we have

$$2\pi i f(x) = \int_{b-i\infty}^{a+i\infty} \frac{f(t)\,dt}{t-x} + \int_{a-i\infty}^{a+i\infty} \frac{f(t)\,dt}{x-t} \quad .$$

We may make use of the expansions already obtained in terms of the $P_n(x)$ and the $Q_n(x)$ and get without difficulty the theorem,

THEOREM V. *Let $f(x)$ be analytic, except, perhaps, at infinity, within and on the boundaries of the strip between the lines $R(x) = a$ and $R(x) = b$, $b > a > 0$. Furthermore, let $f(x)$ have the property that for all values of x in and on the boundaries of this strip there exist positive numbers ϵ and K such that we have the inequality,*

$$|\, x^{\epsilon} f(x) \,| < K,$$

and for x on the line $R(x) = a$ we have the additional inequality,

$$|\, x^{a+\frac{1}{2}+\epsilon}\, e^{|\alpha|\pi/2} f(x) \,| < K.$$

Then for x on the interior of the strip, $f(x)$ admits of the expansions,

$$f(x) = \sum_{n=0}^{\infty} a_n P_n(x) + \sum_{n=0}^{\infty} b_n Q_n(x) = \sum_{n=0}^{\infty} \{a_n P_n(x) + b_n Q_n(x)\},$$

where

$$a_n = 1/2\pi i \int_{b-i\infty}^{b+i\infty} Q_n(t) f(t)\,dt, \qquad b_n = 1/2\pi i \int_{a-i\infty}^{a+i\infty} P_n(t) f(t)\,dt;$$

and

$$f(x) = \sum_{n=0}^{\infty} \alpha_n \mathfrak{P}_n(x) + \sum_{n=0}^{\infty} \beta_n \mathfrak{Q}_n(x) = \sum_{n=0}^{\infty} \{\alpha_n \mathfrak{P}_n(x) + \beta_n \mathfrak{Q}_n(x)\},$$

where

$$\alpha_n = 1/2\pi i \int_{b-i\infty}^{b+i\infty} \mathfrak{Q}_n(t) f(t)\,dt, \quad \beta_n = 1/2\pi i \int_{a-i\infty}^{a+i\infty} \mathfrak{P}_n(t) f(t)\,dt.$$

IV. Expansions of Functions of Several Variables.

16. The theory developed in the preceding sections can be extended without difficulty to the case of any finite number of variables. The theorem

below is stated for the general case of λ variables and, for simplicity, the case of two variables will be treated.

THEOREM VI. *Let* $f(x_1, x_2, \cdots, x_\lambda)$ *be a function of the independent variables,* $x_1, x_2, \cdots, x_\lambda$ *having the properties:*

(A) $f(x_1, x_2, \cdots, x_\lambda)$, *considered as a function of* x_k, *is analytic, except, perhaps, at infinity, in the left half of the* x_k *plane and on the line* $R(x_k) = a_k$, $a_k > 0$, $(k = 1, 2, \cdots, \lambda)$.

(B) $|(x_1, x_2, \cdots, x_\lambda)^\epsilon f(x_1, x_2, \cdots, x_\lambda)| < K$, *for all values of* x_k *on and to the left of the line* $R(x_k) = a$.

Then for x_k *such that* $R(x_k) < a_k$ *we have the valid expansion,*

$$f(x_1, x_2, \cdots, x_\lambda) = \sum_{n_1=0}^{\infty} \sum_{n_2=0}^{\infty} \cdots \sum_{n_\lambda=0}^{\infty} a_{n_1, n_2, \ldots, n_\lambda} P_{n_1}(x_1) P_{n_2}(x_2) \cdots P_{n_\lambda}(x_\lambda),$$

where

$$a_{n_1, n_2, \ldots, n_\lambda} = (1/2\pi i)^\lambda \int_{a_1-i\infty}^{a_1+i\infty} \int_{a_2-i\infty}^{a_2+i\infty} \cdots \int_{a_\lambda-i\infty}^{a_\lambda+i\infty} \{Q_{n_1}(t_1) Q_{n_2}(t_2) \cdots Q_{n_\lambda}(t_\lambda)$$
$$\cdot f(t_1, t_2, \cdots, t_\lambda)\} \, dt_\lambda \cdots dt_2 dt_1.$$

Cauchy's integral theorem for two variables corresponding to the modified form which we have been using may be stated in the form,

$$f(x_1, x_2,) = (1/2\pi i)^2 \int_{a_1-i\infty}^{a_1+i\infty} \int_{a_2-i\infty}^{a_2+i\infty} \sum_{n_1=0}^{\infty} P_{n_1}(x_1) Q_{n_2}(t_2)$$
$$\cdot \sum_{n_2=0}^{\infty} P_{n_2}(x_2) Q_{n_2}(t_2) f(t_1, t_2) \, dt_2 dt_1,$$
$$= \sum_{n_1=0}^{\infty} \sum_{n_2=0}^{\infty} a_{n_1, n_2} P_{n_1}(x_1) P_{n_2}(x_2),$$

where

$$a_{n_1, n_2} = (1/2\pi i)^2 \int_{a_1-i\infty}^{a_1+i\infty} \int_{a_2-i\infty}^{a_2+i\infty} Q_{n_1}(t_1) Q_{n_2}(t_2) f(t_1, t_2) \, dt_2 dt_1,$$

where term by term integration is justified by exactly the same procedure as in the case of one variable. The method obviously applies to the case of any finite number of variables.

There is a like theorem obtained by replacing $P_{n_k}(x_k)$ and $Q_{n_k}(t_k)$ by $\mathfrak{P}_{n_k}(x_k)$, $\mathfrak{Q}_{n_k}(t_k)$ respectively. And it is evident that by making a further restriction analogous to the one in Theorem IV we may state theorems for λ variables corresponding to Theorems IV and V.

On the Value of the Napierian Base.

By Derrick Henry Lehmer.

Four computations of the Napierian base have already been made to more than one hundred decimal places. The first of these was made by William Shanks in 1854.* Using the well-known series, he carried the calculation to 205 decimal places. This value is incorrect beyond the 187 decimal place. The second computation was made in 1871 by J. W. L. Glaisher.† This calculation was carried to only 137 places in order to verify a value given to this extent by Shanks in 1853. In making his calculation, Glaisher used a continued fraction due to Lambert namely:

$$e = 1 + \frac{2}{1+}\,\frac{1}{6+}\,\frac{1}{10+}\,\frac{1}{14+}\cdots\frac{1}{4n-2+}\cdots$$

The third computation was made by J. M. Boorman ‡ in 1884. The value was obtained in this case by taking two terms of the series at a time. The work was carried to 346 decimals of which only 223 are correct. The fourth calculation was made in 1893 by Fr. Tichánek.§ This 225-place value differs from the true value by about a unit in the last place and is therefore more accurate than Boorman's value. Having devised a very effective method of finding the quotient of a pair of large integers, I was induced to undertake the computation of this important constant by means of the above continued fraction. The limit of 707 places of decimals was chosen to match Shanks' last value for π.

We have from the general theory of the continued fraction,

$$e = \frac{A_n}{B_n} + \left[\left(\frac{A_{n+1}}{B_{n+1}} - \frac{A_n}{B_n}\right) + \left(\frac{A_{n+2}}{B_{n+2}} - \frac{A_{n+1}}{B_{n+1}}\right) + \cdots\right]$$

where A_n/B_n is the nth convergent. From this it is seen that the calculation divides itself into three main parts:

(1) The calculation of the numerator and denominator of the nth convergent.

(2) The division of the numerator by the denominator.

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* *Proceedings of the Royal Society of London*, Vol. 6, p. 397.

† *British Association Report*, 1871, pp. 16-18 (Sectional Trans.).

‡ *Mathematical Magazine*, Oct. 1884, p. 204.

§ *Fortschritte der Mathematik*, Vol. 25, p. 736.

(3) The calculation of the expression enclosed in the brackets which we shall call the "correction series." A brief account will be given of the work involved in these three parts together with the processes of checking each part.

The computation of the convergents is accomplished by repeated application of the recursion formulas:

$$A_n = (4n - 6)A_{n-1} + A_{n-2}, \quad B_n = (4n - 6)B_{n-1} + B_{n-2}.$$

In this computation a simple positive check is available. Many curious and important facts exist concerning the A's and B's of the successive convergents taken modulo any prime. The general theory has been given by D. N. Lehmer * and only those properties of use in checking need to be considered here. Taken modulo any odd prime p, the successive numerators and denominators recur with a period of length $2p$. The period for the numerators is divided into two identical parts each of which is palindromic, the centers of the palindromes being at $n = (p + 1)/2$ and $n = (3p + 1)/2$. The same is true for the denominators except that the terms in the second half-period are the negatives of the corresponding terms in the first. This means that if the first $(p + 1)/2$ residues are computed for the numerators and the denominators, all the further ones are known. Now to check any particular A or B it is only necessary to cast out p and compare the remainder with the one predicted. As a precaution different values of p were used. The calculation was carried as far as the 113th numerator and the 153rd denominator corresponding to the partial quotients 446 and 606 respectively. The last two numerators and denominators are given below so that further calculations might start with the 114th numerator and the 154th denominator.

$A_{112} =$ 10 47311 24484 46039 55576 63337 23679 36336 11153 60367 03672
55893 07944 34385 00679 65661 15306 83824 60223 32488 61481
63240 47880 31650 46619 56463 83860 53971 61583 44513 56296
67733 57831 95872 48389 05715 89251 00155 40151 80934 02214
72424 47219 50810 52871 01415 29233 32780 83125 86433

$A_{113} =$
4671 03184 67090 85762 06271 91949 67771 32406 11996 84289 59296
84291 58732 49614 03308 27983 32679 28940 87219 57739 23258
13240 73308 38752 01244 76539 92460 24034 82656 60616 91266
52515 07308 69999 00791 40909 79879 49147 27354 97933 60196
76043 38921 13189 52432 21248 88528 19537 01205 98849

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* *American Journal of Mathematics*, Vol. 40, Oct. 1918, pp. 375-390.

B_{152} = 19551 43005 28544 41378 95432 37087 91389 31560 12537 51465
14516 87840 63923 09013 62522 24501 41387 03498 69753 47012
02498 44145 66851 68818 05120 34702 44263 83004 42421 89149
66865 46759 95816 98371 16694 23731 91594 41745 84049 29272
56563 27398 04760 64786 42330 51079 70062 08770 34727 85256
07903 27801 07132 63974 01871 91577 01989 11616 77970 73782
96834 06627 44284 51695 16976 17753 21721 41455 85860 87802
97551

B_{153}=118 48199 08939 81353 74029 83433 08016 00755 31667 05910 81282
67671 78454 02997 17099 58748 65477 86838 06635 89732 69182
45858 00061 15252 97602 00912 27708 18649 30794 29891 36046
36149 55211 44225 30142 11719 28058 72563 44745 44104 33351
79817 65666 11212 45922 67993 28735 07391 65094 82510 57968
07444 17697 72653 98928 52783 38928 87663 40398 64346 46435
55530 01106 84155 38432 55945 86205 66687 10069 83395 83891
67057

In the second part of the calculation—that of dividing A_n by B_n—a large amount of actual division was saved by multiplying B_{113} by Shanks' value for e as far as it is known to be correct, and subtracting the product from A_{113}. The remainder R was then divided by B_{113} and the quotient annexed to Shanks' value. The multiplication was performed according to a method given by D. N. Lehmer * and each partial product as well as the complete product was checked by casting out 1001. The details of the division process are too complicated to be given here in full. Briefly, the process is a reversal of cross-multiplication, and a number Q is sought such that $B_{113}Q = R$. Q is determined period by period. The division was checked step by step by casting out 1001 and was carried as far as 522 decimals. This quotient, when annexed to Shanks' 187 correct figures gives A_{113}/B_{113} correct to 709 decimal places. Although this is probably the largest division ever made, it required the setting down of less than 4,000 figures, the calculation being made on a ten-place computing machine in twenty-four hours. The effectiveness of the new process is more apparent when one compares it with Glaisher's 137-place division which required some 18,000 figures.

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* *American Mathematical Monthly*, Vol. 30, Feb. 1923, p. 67.

The evaluation of the correction series:

$$S_n = \left(\frac{A_{n+1}}{B_{n+1}} - \frac{A_n}{B_n}\right) + \left(\frac{A_{n+2}}{B_{n+2}} - \frac{A_{n+1}}{B_{n+1}}\right) + \cdots$$

was greatly simplified by writing it in the following form:

$$S_n = (-1)^{n+1} 2 \left(\frac{4n+2}{B_n B_{n+2}} + \frac{4n+10}{B_{n+2} B_{n+4}} + \cdots\right)$$

which is easily obtained by using the relations:

$$A_{n+1} B_n - A_n B_{n+1} = (-1)^{n+1} 2,$$
$$B_n = (4n-6) B_{n-1} + B_{n-2}.$$

This gives for $n = 113$

$$S_{113} = 2 \left(\frac{454}{B_{113} B_{115}} + \frac{462}{B_{115} B_{117}} + \frac{470}{B_{117} B_{119}} + \cdots\right).$$

Twenty terms of this highly convergent series are sufficient to give the following value which is correct to 709 decimals.

$S_{113} = .0^{498}15051$ 38620 54024 41229 36919 32265 04583 02311 17438 16742
30918 03984 62393 15689 43133 33053 40503 92111 75295 64524
36677 37817 45122 94622 37124 40008 30190 23669 41230 24313
33649 14353 06143 18243 51063 48246 24696 95290 11384 13873
62161 69335 5

Every step in the calculation of S_{113} was checked by casting out 1001. Adding this value to the value obtained for A_{113}/B_{113}, we get the following value of the Napierian base:

$e = 2.71828$ 18284 59045 23536 02874 71352 66249 77572 47093 69995
95749 66967 62772 40766 30353 54759 45713 82178 52516 64274
27466 39193 20030 59921 81741 35966 29043 57290 03342 95260
59563 07381 32328 62794 34907 63233 82988 07531 95251 01901
15738 34187 93070 21540 89149 93488 41675 09244 76146 06680
82264 80016 84774 11853 74234 54424 37107 53907 77449 92069
55170 27618 38606 26133 13845 83000 75204 49338 26560 29760
67371 13200 70932 87091 27443 74704 72306 96977 20931 01416
92836 81902 55151 08657 46377 21112 52389 78442 50569 53696
77078 54499 69967 94686 44549 05987 93163 68892 30098 79312
77361 78215 42499 92295 76351 48220 82698 95193 66803 31825
28869 39849 64651 05820 93923 98294 88793 32036 25094 43117
30123 81970 68416 14039 70198 37679 32068 32823 76464 80429
53118 02328 78250 98194 55815 30175 67173 61332 06981 12509
96181 88

The above value for e has been subjected to the following independent verification. If the value for A_{153}/B_{153} were found it would represent e correctly to 707 places without using S_{153}. Therefore if the above value for e were multiplied by B_{153} the product should give A_{153} followed by 349 zeros since A_{153} is a 358-digit number. If it can be shown that the last few of these zeros actually appear, it serves as a check on those digits of the value for e which are responsible for the appearance of these zeros, namely the last 350 digits. Now by cross-multiplication, we can easily find that period in the product which lies under the last period of the value for e. This was actually done, using a 709-place value of e, and the period in the product, which is of the same order as the last nine figures of e, was found to be 999999999 which amounts to 000000000. This then is a verification of the last 350 digits; but it is easily seen from the way in which the work was carried out that the last half of the value is absolutely dependent upon the first half and therefore this check is a verification of the entire value. Similar checks using larger denominators can easily be made. Moreover, if we wish to determine the digits between the 707th and the 722nd places for instance, we have only to calculate the 154th denominator and multiply it by the above value of e in such a way as to exhibit that part of the product which lies under the digits in question. We then write in those digits which will make this part of the product a series of 9's. Continuing this process and allowing the steps to overlap slightly, we can find as many digits beyond the 707th place as we desire and have at the same time a rigorous check on the calculation.

On Differential Inversive Geometry.

By Frank Morley.

§ 1. *The Integral Invariant.*

Through the Columbia dissertation of G. W. Mullins, "Differential Invariants under the Inversion Groups," and the memoirs of Liebmann (*Munich Berichte,* 1923) and Kubota (*Tohuku University Reports,* 1923?), the foundations of differential inversive geometry for a plane curve are established. But the following self-contained mode of approach seems desirable.

Denote the Schwarzian derivative of x as to y by $\{x, y\}$. Then we have for any number n of related numbers $x, y, \cdots, l$. Cayley's * cyclic formula

$$C_n\colon\ \{x,y\}(dy)^2+\{y,z\}(dz)^2+\cdots+\{l,x\}(dx)^2=0.$$

Let the curve in question be given by a self-conjugate equation in x and its conjugate $\bar{x}$. Then by C_2

$$\{x,\bar{x}\}(d\bar{x})^2+\{\bar{x},x\}(dx)^2=0,$$

so that $\{x,\bar{x}\}(d\bar{x})^2$ is an imaginary, say

$$\text{1)}\qquad \{x,\bar{x}\}(d\bar{x})^2=\pm 2\iota(d\lambda)^2,\ \text{ where } \lambda \text{ is real.}$$

Apply a homography $x=\dfrac{ay+b}{cy+d}$ and use Cayley's rule for the cycle $x, \bar{x}, \bar{y}, y$. Then since $\{\bar{x},\bar{y}\}=0$ and $\{y,x\}=0$,

$$\begin{aligned}\{x,\bar{x}\}(d\bar{x})^2&=-\{\bar{y},y\}dy^2\\&=\{y,\bar{y}\}d\bar{y}^2.\end{aligned}$$

Hence λ is an invariant under homographies and is the proper real parameter.

Using C_3 for the cycle $\bar{x}, \lambda, x$

$$\{\bar{x},d\}(d\lambda)^2+\{\lambda,x\}(dx)^2+\{x,\bar{x}\}d\bar{x}^2=0,$$

so that $\qquad \{x,\lambda\}-\{\bar{x},\lambda\}=\pm 2\iota.$

Hence $\qquad \{x,\lambda\}=\pm\iota+I,$ when I is real.

I is then the fundamental differential invariant.

* Cayley, *Camb. Phil. Trans.*, Vol. 13, 1880.

If the curve is given in the form $x=f(\mu)$ where μ is any real parameter, then from C_3

$$(d\lambda/d\mu)^2\{x,\lambda\}=\{x,\mu\}-\{\lambda,\mu\},$$

so that
$$\pm 2\iota(d\lambda/d\mu)^2=\{x,\mu\}-\{\bar{x},\mu\}$$

and
$$2I(d\lambda/d\mu)^2=\{x,\mu\}+\{\bar{x},\mu\}-2\{\lambda,\mu\}.$$

§2. *The Curve whose Intrinsic Equation is Linear.*

The intrinsic homographic equation of a curve is the relation of I to λ. It remains the same for all the homographic transformations of the curve. It is known that the curve

$$I=\text{const.}$$

is the loxodrome, or isogonal trajectory of arcs of circles from a point to another. We seek the curve given by

$$I=a\lambda+b$$

or taking a proper initial point on the curve from which to measure λ, namely a point such that at it $I=0$ (or the closest loxodrome has the angle $\pi/4$).

$$I=a\lambda.$$

The curve is given by the equation

$$\{x,\lambda\}=a(\lambda\pm\iota/a).$$

Whence x is a ratio of two solutions of

$$(d^2v/d\lambda^2)+\tfrac{1}{2}va(\lambda\pm\iota/a).$$

(Forsyth, *Treatise on Dif. Equations*, § 61).
Writing $\lambda\pm\iota/a=\kappa\mu$, where $\kappa^3a=2$, this equation becames

$$d^2v/d\mu^2+v\mu=0$$

and has the solutions in power-series

$$v_1=\mu-2\mu^4/4!+2.5.\mu^7/7!-\cdots$$
$$v_2=1-\mu^3/3!+4\mu^6/6!-4.7.\mu^9/9!+\cdots$$

The curve is then $x=v_1/v_2$, where μ moves parallel to the axis of reals. To see what happens when μ is large, we note that our series are expressions for Bessel functions; apart from a constant factor, v_1 is $J_{1/3}(\tfrac{2}{3}\mu^{3/2})$ and

v_2 is $J_{-1/3}(\tfrac{2}{3}\mu^{3/2})$. (Forsyth, *loc. cit.*, § 101 and 111). Now the development of a Bessel function in negative powers of the argument was given by Lommel. (Forsyth, *loc. cit.*, § 105).

Applying Lommel's formula we have for z large to a first approximation

$$J_{1/3}(z) = (2/\pi z)^{1/2} \cos(z - \pi/4 - \pi/6) + \cdots$$

$$J_{1/3}(z) = (2/\pi z)^{1/2} \cos(z - \pi/4 + \pi/6) + \cdots$$

Thus x is to a first approximation homographic with $\exp 2\iota z$, where $z = \tfrac{2}{3}\mu^{3/2}$.

The general question of passing from the intrinsic equation of a curve, $I = f(\lambda)$ to the map equation $x = f(\lambda)$ is that of integrating

$$\{x, \lambda\} = f(\lambda) \pm \iota.$$

Thus it comes under the method sketched by Klein, "Ueber Gewisse Differentialgleichungen dritter Ordnung," *Math. Ann.*, Vol. 23 or *Works,* Vol. 3, p. 721.